MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 2 | **MATHEMATICS 1000** | FALL 2023

SOLUTIONS

[4] 1. (a) This is a quasirational function for which direct substitution yields a $\frac{0}{0}$ indeterminate form, so we use the Rationalisation Method. There is a radical in *both* the numerator and the denominator, so let's first try rationalising the numerator:

$$\lim_{x\to-2}\frac{\sqrt{2-x}-2}{3-\sqrt{4x+17}}\cdot\frac{\sqrt{2-x}+2}{\sqrt{2-x}+2}=\lim_{x\to-2}\frac{(2-x)-4}{\left(3-\sqrt{4x+17}\right)\left(\sqrt{2-x}+2\right)}$$
$$=\lim_{x\to-2}\frac{-2-x}{\left(3-\sqrt{4x+17}\right)\left(\sqrt{2-x}+2\right)}.$$

Now we'll rationalise the denominator:

$$\lim_{x\to-2}\frac{\sqrt{2-x}-2}{3-\sqrt{4x+17}}=\lim_{x\to-2}\frac{-2-x}{\left(3-\sqrt{4x+17}\right)\left(\sqrt{2-x}+2\right)}\cdot\frac{3+\sqrt{4x+17}}{3+\sqrt{4x+17}}$$
$$=\lim_{x\to-2}\frac{(-2-x)\left(3+\sqrt{4x+17}\right)}{[9-(4x+17)]\left(\sqrt{2-x}+2\right)}$$
$$=\lim_{x\to-2}\frac{(-2-x)\left(3+\sqrt{4x+17}\right)}{(-4x-8)\left(\sqrt{2-x}+2\right)}$$
$$=\lim_{x\to-2}\frac{-(x+2)\left(3+\sqrt{4x+17}\right)}{-4(x+2)\left(\sqrt{2-x}+2\right)}$$
$$=\lim_{x\to-2}\frac{-\left(3+\sqrt{4x+17}\right)}{-4\left(\sqrt{2-x}+2\right)}$$
$$=\frac{-(3+3)}{-4(2+2)}$$
$$=\frac{3}{8}.$$

[4] (b) Direct substitution yields another type of indeterminate form ($\infty-\infty$) so we first need to rewrite the given function in a way that will allow us to use the techniques we've learned. We have

$$\lim_{t\to5}[t(t^2-25)^{-1}-(t^2-8t+15)^{-1}]=\lim_{t\to5}\left[\frac{t}{t^2-25}-\frac{1}{t^2-8t+15}\right]$$
$$=\lim_{t\to5}\left[\frac{t}{(t-5)(t+5)}-\frac{1}{(t-5)(t-3)}\right]$$
$$=\lim_{t\to5}\frac{t(t-3)-(t+5)}{(t-5)(t+5)(t-3)}$$
$$=\lim_{t\to5}\frac{t^2-4t-5}{(t-5)(t+5)(t-3)}.$$

Now we've obtained a rational function (and note that direct substitution produces a $\frac{0}{0}$ indeterminate form) so we can use the Cancellation Method:

$$\begin{aligned}\lim_{t\to 5}\frac{t^2-4t-5}{(t-5)(t+5)(t-3)} &= \lim_{t\to 5}\frac{(t-5)(t+1)}{(t-5)(t+5)(t-3)}\\ &= \lim_{t\to 5}\frac{t+1}{(t+5)(t-3)}\\ &= \frac{6}{10\cdot 2}\\ &= \frac{3}{10}.\end{aligned}$$

[3] (c) Again, direct substitution results in a $\frac{0}{0}$ indeterminate form. But recall that, for any θ,

$$1-\cos^2(\theta)=\sin^2(\theta).$$

This means that we can rewrite the given limit as

$$\lim_{x\to 0}\frac{\sin^2(x)}{\sin^2(4x)}.$$

Now we can try using the special trigonometric limit

$$\lim_{x\to 0}\frac{\sin(x)}{x}=1 \quad \text{or} \quad \lim_{x\to 0}\frac{x}{\sin(x)}=1.$$

First let's concentrate on the two sine functions in the numerator. In order to use the special limit, we need an x in the denominator for each of them, so we multiply the numerator and the denominator by x^2:

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^2(x)}{\sin^2(4x)}\cdot\frac{x^2}{x^2} &= \lim_{x\to 0}\frac{\sin^2(x)}{x^2}\cdot\lim_{x\to 0}\frac{x^2}{\sin^2(4x)}\\ &= \lim_{x\to 0}\frac{\sin(x)}{x}\cdot\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\lim_{x\to 0}\frac{x^2}{\sin^2(4x)}\\ &= 1\cdot 1\cdot\lim_{x\to 0}\frac{x^2}{\sin^2(4x)}\\ &= \lim_{x\to 0}\frac{x^2}{\sin^2(4x)}.\end{aligned}$$

Now, for the remaining limit, we need a $4x$ in the numerator for each of the two sine functions in the denominator. We've already got an x^2 there from our previous step, so we just multiple the numerator and denominator by $4^2=16$:

$$\lim_{x\to 0}\frac{x^2}{\sin^2(4x)}\cdot\frac{16}{16}=\frac{1}{16}\lim_{x\to 0}\frac{4x}{\sin(4x)}\cdot\lim_{x\to 0}\frac{4x}{\sin(4x)}.$$

Note that as $x \to 0$, $4x \to 0$ as well, so these limits are in the same form as the special limit. Finally, then, we have

$$\lim_{x\to 0} \frac{1-\cos^2(x)}{1-\cos^2(4x)} = \frac{1}{16} \cdot 1 \cdot 1 = \frac{1}{16}.$$

[3] 2. Since this is a piecewise function whose behaviour changes at $x = 4$, we must check the one-sided limits:

$$\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} (2x - k) = 8 - k$$

and

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^+} (x+k)^2 = (4+k)^2 = k^2 + 8k + 16.$$

If the limit exists, then these one-sided limits must be equal, so we set

$$\begin{aligned} 8 - k &= k^2 + 8k + 16 \\ k^2 + 9k + 8 &= 0 \\ (k+8)(k+1) &= 0, \end{aligned}$$

and hence $k = -8$ or $k = -1$.

(Note that the value of $f(x)$ at $x = 4$ did not affect our workings, because the limit considers the behaviour of the function *near* $x = 4$, but not *at* $x = 4$.)

[6] 3. First we set

$$\begin{aligned} x^4 - 4x^3 + 4x^2 &= 0 \\ x^2(x^2 - 4x + 4) &= 0 \\ x^2(x-2)^2 &= 0, \end{aligned}$$

so the possible vertical asymptotes are $x = 0$ and $x = 2$.

At $x = 0$, the numerator is 0 as well, so we need to check the limit:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{30x^2 - 5x^4 - 5x^3}{x^4 - 4x^3 + 4x^2} = \lim_{x\to 0} \frac{-5x^2(x^2 + x - 6)}{x^2(x-2)^2} = \lim_{x\to 0} \frac{-5(x^2+x-6)}{(x-2)^2} = \frac{15}{2}.$$

Since the limit exists, we can conclude that $x = 0$ is not a vertical asymptote.

At $x = 2$, the numerator is also 0, and so we compute

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{30x^2 - 5x^4 - 5x^3}{x^4 - 4x^3 + 4x^2} = \lim_{x\to 2} \frac{-5x^2(x-2)(x+3)}{x^2(x-2)^2} = \lim_{x\to 2} \frac{-5(x+3)}{x-2}.$$

Now direct substitution results in a $\frac{-25}{0}$ form, so $x = 2$ is a vertical asymptote.

To determine the one-sided limits of $f(x)$ as $x \to 2$, we consider the expression $\dfrac{-5(x+3)}{x-2}$. Near $x = 2$, the numerator is approximately $-5 \cdot 5 = -25$. From the left as $x \to 2$, $x - 2$ is a small negative number, and so $\dfrac{-5(x+3)}{x-2}$ becomes a large positive number. Hence

$$\lim_{x \to 2^-} f(x) = \infty.$$

On the other hand, from the right as $x \to 2$, $x - 2$ is a small positive number, and therefore $\dfrac{-5(x+3)}{x-2}$ becomes a large negative number. In other words,

$$\lim_{x \to 2^+} f(x) = -\infty.$$