MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

Section 1.7	**Math 1000 Worksheet**	Fall 2023

SOLUTIONS

1. (a) The only value of x for which either definition of $f(x)$ is undefined is at $x = 2$, which is also the point at which the function definition changes. Here, $f(2) = 0$. We do not need to evaluate the one-sided limits because $f(x)$ is defined in the same way to either side of $x = 2$. We simply have

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

However, this means that $f(2) \neq \lim_{x\to 2} f(x)$. Thus $x = 2$ is a removable discontinuity.

(b) First we identify any points at which the function will not be defined. This can only happen when the first part of the definition has a zero denominator, and since

$$\frac{x+1}{x^2-x-2} = \frac{x+1}{(x-2)(x+1)}$$

this occurs for $x = 2$ and $x = -1$. However, $f(x)$ only adopts this definition for $x < 1$, so only $x = -1$ is a discontinuity. To classify it, we need to take the limit at $x \to -1$. Since the numerator is also zero when $x = -1$, we have a $\frac{0}{0}$ indeterminate form. Hence we use the cancellation method:

$$\lim_{x\to -1} \frac{x+1}{x^2-x-2} = \lim_{x\to -1} \frac{x+1}{(x-2)(x+1)} = \lim_{x\to -1} \frac{1}{x-2} = -\frac{1}{3}.$$

Since the limit exists, $x = -1$ is a removable discontinuity.

We also need to check the points where the definition of $f(x)$ changes; the only such point is $x = 1$. We have $f(1) = 2$. Also,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x+1}{x^2-x-2} = \lim_{x\to 1^-} \frac{x+1}{(x-2)(x+1)} = \lim_{x\to 1^-} \frac{1}{x-2} = -1$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3-x^2) = 2.$$

Hence the limit as $x \to 1$ does not exist, and so $x = 1$ is a discontinuity. Since the one-sided limits both exist, however, $x = 1$ is a jump discontinuity.

(c) First, note that $\dfrac{x}{x^2-5x}$ is undefined if

$$x^2 - 5x = x(x-5) = 0.$$

This occurs when $x = 0$ or $x = 5$, but we ignore the second possibility because this definition only applies for $x < 1$. At $x = 0$, direct substitution gives a $\frac{0}{0}$ indeterminate form, so we take the limit using the cancellation method:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{x^2-5x} = \lim_{x\to 0} \frac{x}{x(x-5)} = \lim_{x\to 0} \frac{1}{x-5} = \frac{1}{0-5} = -\frac{1}{5}.$$

Since the limit exists, there is a removable discontinuity at $x = 0$.

Next, observe that $\dfrac{2}{x-9}$ is undefined if $x - 9 = 0$, that is, for $x = 9$. Direct substitution produces a $\frac{K}{0}$ form, so this is a vertical asymptote and we can conclude that $\lim_{x\to 9} f(x)$ does not exist. Thus there is an infinite discontinuity at $x = 9$.

Lastly, we must consider $x = 1$, since this is where the definition of the piecewise function changes. First, note that

$$f(1) = \frac{2}{1-9} = -\frac{1}{4}.$$

Because $f(x)$ is defined differently to the left and to the right of $x = 1$, we evaluate the one-sided limits:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x}{x^2-5x} = \frac{1}{1-5} = -\frac{1}{4}$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{2}{x-9} = \frac{2}{1-9} = -\frac{1}{4}.$$

Hence

$$\lim_{x\to 1} f(x) = -\frac{1}{4} = f(1),$$

and so $x = 1$ is not a discontinuity at all.

2. Observe that $f(x)$ is a continuous function (since it is a polynomial), and $f(-2) = 59$ while $f(2) = -21$. By the Intermediate Value Theorem, there must be at least one x on the interval $[-2, 2]$ for which $f(x) = 0$, which means that there must be a root of $f(x)$ on that interval.