MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 1.6 **Math 1000 Worksheet** FALL 2025

SOLUTIONS

1. (a) First, we observe that

$$f(-2) = \frac{(-2)^2+4}{2(-2)^2+4} = \frac{8}{12} = \frac{2}{3},$$

so $f(-2)$ is defined. Next, we evaluate

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{x^2+4}{2x^2+4} = \frac{2}{3}$$

and

$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \frac{x^2-4}{2x+4} = \lim_{x\to -2^+} \frac{(x-2)(x+2)}{2(x+2)} = \lim_{x\to -2^+} \frac{x-2}{2} = -2.$$

Since the one-sided limits are not equal, $\lim_{x\to -2} f(x)$ does not exist. Hence $f(x)$ is not continuous at $x=-2$, and it is a non-removable discontinuity.

(b) First, we observe that

$$f(1) = \frac{1^2-4}{2\cdot 1+4} = \frac{-3}{6} = -\frac{1}{2}.$$

Next, we evaluate

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2-4}{2x+4} = -\frac{1}{2}$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^2-4}{x^2-9x+14} = \frac{1^2-4}{1^2-9\cdot 1+14} = -\frac{3}{6} = -\frac{1}{2}.$$

This time, the one-sided limits are equal, so

$$\lim_{x\to 1} f(x) = -1 = f(1).$$

Hence all three parts of the definition of continuity at a point are satisfied, and so $f(x)$ is continuous at $x=1$.

(c) Since direct substitution of $x=2$ produces a $\frac{0}{0}$ form, we know that $f(2)$ is undefined, but we must apply the Cancellation Method to determine whether the limit exists:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x^2-9x+14} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x-7)} = \lim_{x\to 2} \frac{x+2}{x-7} = -\frac{4}{5}.$$

Since the limit exists, the discontinuity is removable.

2. We have $f(1) = 2k + 3$, which is defined for all k. By cancellation,

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2 + (k-1)x - k}{x - 1} = \lim_{x\to 1} \frac{(x-1)(x+k)}{x-1} = \lim_{x\to 1}(x+k) = 1 + k,$$

so the limit exists for all k. In order for $f(x)$ to satisfy the requirement that $\lim_{x\to 1} f(x) = f(1)$, we set

$$1 + k = 2k + 3 \implies k = -2,$$

so $f(x)$ is continuous at $x = 1$ only if $k = -2.$

3. First observe that $f(2) = 2k^2 - 5$, which is defined for all k. Since $f(x)$ is a piecewise function whose definition changes at $x = 2$, we investigate the one-sided limits:

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{1}{x-4} = -\frac{1}{2} \quad \text{and} \quad \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (k^2 x - 5) = 2k^2 - 5.$$

For the one-sided limits to be equal, we set $2k^2 - 5 = -\frac{1}{2}$ and hence $k = \pm\frac{3}{2}$. Note that for either value of k,

$$f(2) = \lim_{x\to 2} f(x) = -\frac{1}{2},$$

so $f(x)$ is continuous at $x = 2$ for $k = \frac{3}{2}$ and $k = -\frac{3}{2}.$

4. First note that $f(0) = k + \frac{5}{6}$, which is defined for any k. Next,

$$\begin{aligned}\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sqrt{kx^2+1} - 1}{3x^2} &= \lim_{x\to 0} \frac{\sqrt{kx^2+1} - 1}{3x^2} \cdot \frac{\sqrt{kx^2+1} + 1}{\sqrt{kx^2+1} + 1} \\ &= \lim_{x\to 0} \frac{kx^2}{3x^2\left(\sqrt{kx^2+1} + 1\right)} = \lim_{x\to 0} \frac{k}{3\left(\sqrt{kx^2+1} + 1\right)} = \frac{k}{6},\end{aligned}$$

so the limit exists for any k. Finally, we need to determine when $f(0) = \lim_{x\to 0} f(x)$. We set

$$k + \frac{5}{6} = \frac{k}{6} \implies \frac{5k}{6} = -\frac{5}{6} \implies k = -1,$$

so the only value of k which makes $f(x)$ continuous at $x = 0$ is $k = -1.$

5. (a) The only value of x for which either definition of $f(x)$ is undefined is at $x = 2$, which is also the point at which the function definition changes. Here, $f(2) = 0$. We do not need to evaluate the one-sided limits because $f(x)$ is defined in the same way to either side of $x = 2$. We simply have

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 4.$$

However, this means that $f(2) \neq \lim_{x\to 2} f(x)$. Thus $x = 2$ is a removable discontinuity.

(b) First we identify any points at which the function will not be defined. This can only happen when the first part of the definition has a zero denominator, and since

$$\frac{x+1}{x^2-x-2} = \frac{x+1}{(x-2)(x+1)}$$

this occurs for $x = 2$ and $x = -1$. However, $f(x)$ only adopts this definition for $x < 1$, so only $x = -1$ is a discontinuity. To classify it, we need to take the limit at $x \to -1$. Since the numerator is also zero when $x = -1$, we have a $\frac{0}{0}$ indeterminate form. Hence we use the cancellation method:

$$\lim_{x\to -1} \frac{x+1}{x^2-x-2} = \lim_{x\to -1} \frac{x+1}{(x-2)(x+1)} = \lim_{x\to -1} \frac{1}{x-2} = -\frac{1}{3}.$$

Since the limit exists, $x = -1$ is a removable discontinuity.

We also need to check the points where the definition of $f(x)$ changes; the only such point is $x = 1$. We have $f(1) = 2$. Also,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x+1}{x^2-x-2} = \lim_{x\to 1^-} \frac{x+1}{(x-2)(x+1)} = \lim_{x\to 1^-} \frac{1}{x-2} = -1$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (3-x^2) = 2.$$

Hence the limit as $x \to 1$ does not exist, and so $x = 1$ is a discontinuity. Since the one-sided limits both exist, however, $x = 1$ is a jump discontinuity.

(c) First, note that $\dfrac{x}{x^2-5x}$ is undefined if

$$x^2 - 5x = x(x-5) = 0.$$

This occurs when $x = 0$ or $x = 5$, but we ignore the second possibility because this definition only applies for $x < 1$. At $x = 0$, direct substitution gives a $\frac{0}{0}$ indeterminate form, so we take the limit using the cancellation method:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{x^2-5x} = \lim_{x\to 0} \frac{x}{x(x-5)} = \lim_{x\to 0} \frac{1}{x-5} = \frac{1}{0-5} = -\frac{1}{5}.$$

Since the limit exists, there is a removable discontinuity at $x = 0$.

Next, observe that $\dfrac{2}{x-9}$ is undefined if $x - 9 = 0$, that is, for $x = 9$. Direct substitution produces a $\frac{K}{0}$ form, so this is a vertical asymptote and we can conclude that $\lim\limits_{x\to 9} f(x)$ does not exist. Thus there is an infinite discontinuity at $x = 9$.

Lastly, we must consider $x = 1$, since this is where the definition of the piecewise function changes. First, note that

$$f(1) = \frac{2}{1-9} = -\frac{1}{4}.$$

Because $f(x)$ is defined differently to the left and to the right of $x = 1$, we evaluate the one-sided limits:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x}{x^2 - 5x} = \frac{1}{1-5} = -\frac{1}{4}$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{2}{x-9} = \frac{2}{1-9} = -\frac{1}{4}.$$

Hence

$$\lim_{x\to 1} f(x) = -\frac{1}{4} = f(1),$$

and so $x = 1$ is not a discontinuity at all.

6. Observe that $f(x)$ is a continuous function (since it is a polynomial), and $f(-2) = 59$ while $f(2) = -21$. By the Intermediate Value Theorem, there must be at least one x on the interval $[-2, 2]$ for which $f(x) = 0$, which means that there must be a root of $f(x)$ on that interval.