Section 4.4: Absolute Extrema

<u>Def'n</u>: A function $f(x)$ defined on an interval I has an <u>absolute</u> (or global) <u>maximum</u> at $x=p$ if $f(p) \geq f(x)$ for all x in I. It has an <u>absolute</u> (or global) <u>minimum</u> at $x=p$ if $f(p) \leq f(x)$ for all x in I.

The value of $f(x)$ at an absolute maximum is the <u>maximum value</u>. Its value at an absolute minimum is the <u>minimum value</u>. These are both unique.

<u>The Extreme Value Theorem</u>

If a function $f(x)$ is continuous on a closed interval $a \leq x \leq b$ then $f(x)$ possesses both a maximum value and a minimum value on that interval.

Given $f(x)$ which is continuous on $a \le x \le b$, we identify the maximum and minimum values as follows:

1. find all the critical points of $f(x)$ on the interval $a < x < b$
2. evaluate $f(x)$ at each critical point
3. evaluate $f(a)$ and $f(b)$

The greatest of these values is the maximum value, and the least is the minimum value.

eg Find the maximum and minimum values of

$$f(x) = \frac{8}{x^2+3}$$

on $-1 \leq x \leq 1$.

We have $f'(x) = \dfrac{-16x}{(x^2+3)^2}$ which is never

undefined, so we set

$$f'(x) = 0$$

$$-16x = 0 \rightarrow x = 0 \quad \text{(CRITICAL POINT)}$$

$f(0) = \frac{8}{3}$ $\quad$ $f(-1) = 2$ $\quad$ $f(1) = 2$

Thus $\underline{\frac{8}{3}}$ is the maximum value and $\underline{2}$ is the minimum value.

<u>Theorem: The Second Derivative Test</u>

Let $x=p$ be the only critical point of a function $f(x)$ on an open interval I. If $f(x)$, $f'(x)$ and $f''(x)$ are all continuous on I then

1. if $f''(p) < 0$ then $x=p$ is the absolute maximum of $f(x)$ on I
2. if $f''(p) > 0$ then $x=p$ is the absolute minimum of $f(x)$ on I

eg Determine the maximum value of

$$f(x) = \frac{x}{x^2+1}$$

on $0 < x < 5$.

We have $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$ so we set

$$f'(x) = 0$$

$$1 - x^2 = 0 \rightarrow x = -1,\ x = 1$$

(CRITICAL POINTS)

We can omit $x=-1$ because it does not lie in the indicated interval.

Now $f''(x) = \dfrac{2x(x^2-3)}{(x^2+1)^3}$

$$f''(1) = \frac{2(-2)}{8} = -\frac{1}{2} < 0$$

Hence $x=1$ is the absolute maximum by the Second Derivative Test and so the maximum value is $\boxed{f(1) = \frac{1}{2}.}$

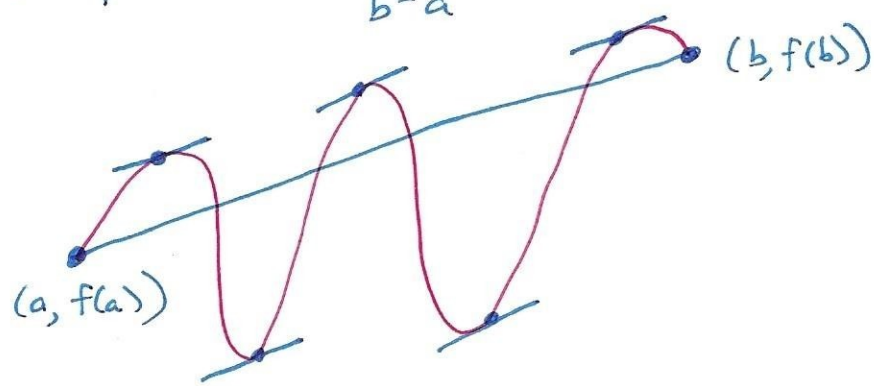
The Mean Value Theorem

If $f(x)$ is a function such that

① $f(x)$ is continuous on $a \le x \le b$, and

② $f(x)$ is differentiable on $a < x < b$

then there is at least one point $x = p$ in $a < x < b$ for which $f'(p) = \dfrac{f(b) - f(a)}{b-a}$.



eg An unknown function $f(x)$ has $f(1) = -2$ and $f'(x) \ge 3$ for all x. What is the minimum possible value of $f(6)$?

Consider the interval $1 \le x \le 6$. We are given that $f'(x)$ for all x, and therefore on $1 < x < 6$ we have $f(x)$ differentiable, while on $1 \le x \le 6$ we have $f(x)$ continuous by virtue of the result that differentiability implies continuity. Thus the Mean Value Theorem applies to $f(x)$.

Now we know that there exists a point $x=p$ where

$$f'(p) = \frac{f(6) - f(1)}{6-1}$$

$$= \frac{f(6) + 2}{5}$$

But since $f'(x) \geq 3$ for all x,

$$\frac{f(6)+2}{5} \geq 3$$

$$f(6) + 2 \geq 15 \rightarrow \boxed{f(6) \geq 13}$$