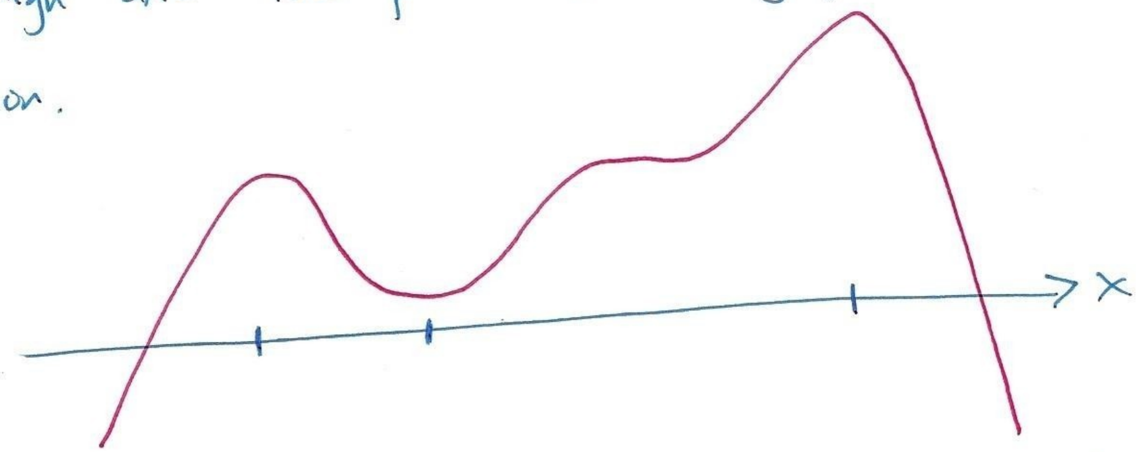
Section 4.2: Relative Extrema and Points of Inflection

Given a function $y = f(x)$, we often want to understand the location and characteristics of its extrema: the high and low points in the graph of the function.



One type of extrema is called absolute or global extrema, and represents the highest and lowest points (if any) anywhere in the graph.

The other type is called relative or local extrema which are the peaks and troughs in the graph.

Def'n: A function $f(x)$ has a relative maximum at $x = p$ if $f(p) \geq f(x)$ for all x in an open interval containing $x = p$. It has a relative minimum at $x = p$ if $f(p) \leq f(x)$ for all x in an open interval containing $x = p$.

Given a function $f(x)$, we want to find and classify any relative extrema.

Relative maxima can only appear in the following forms:

HORIZONTAL TANGENT LINE

NON-DIFFERENTIABLE

and similarly for relative minima.

<u>Fermat's Theorem</u>: If a function $f(x)$ has a relative extremum at $x=p$ then either $f'(p)=0$ or $f'(p)$ is undefined.

<u>Def'n</u>: If $x=p$ is in the domain of $f(x)$ and either $f'(p)=0$ or $f'(p)$ is undefined then we say that $x=p$ is a <u>critical point</u> of $f(x)$.

However, not all critical points are relative extrema. Some are <u>saddle points</u>, which are neither relative maxima nor relative minima.

eg Find the critical points of

$$f(x) = \frac{x^2+3}{x^2-1}.$$

$$f'(x) = \frac{2x(x^2-1) - (x^2+3)\cdot 2x}{(x^2-1)^2} = \frac{-8x}{(x^2-1)^2}$$

Here, $f'(x)$ is undefined if

$$(x^2-1)^2 = 0$$

$$x^2 = 1 \rightarrow x = \pm 1$$

but neither $x=1$ nor $x=-1$ is in the domain of $f(x)$, so they can't be critical points.

Now we set $f'(x) = 0$

$$-8x = 0 \rightarrow x = 0$$

$\boxed{\text{Thus } x=0 \text{ is the only critical point}}$

Def'n: Consider a function $f(x)$ defined on an interval I. Let $x=a$ and $x=b$ be any two points on I such that $a<b$. Then $f(x)$ is increasing on I if $f(a)<f(b)$ for any a and b. Likewise, $f(x)$ is decreasing on I if $f(a)>f(b)$ for any a and b.

Theorem: Given a function $f(x)$ defined on an interval I,

1. if $f'(x)>0$ for all x in I then $f(x)$ is increasing on I
2. if $f'(x)<0$ for all x in I then $f(x)$ is decreasing on I

Then, to determine the intervals on which $f(x)$ is increasing or decreasing, we do the following:

1. find all the critical points of $f(x)$
2. use these critical points to split the domain of $f(x)$ into intervals
3. choose any point in each interval and use it to determine the sign of $f'(x)$, and thus whether $f(x)$ is increasing or decreasing on that interval.

eg Determine the intervals where

$$f(x) = x^3 - x$$

is increasing or decreasing.

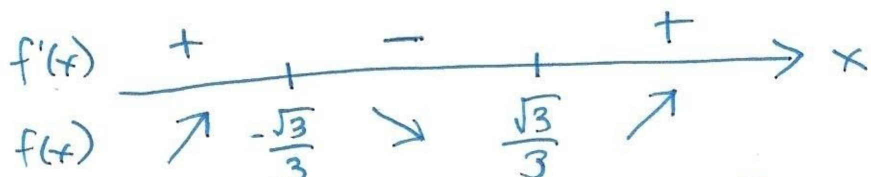
$$f'(x) = 3x^2 - 1$$

Since $f'(x)$ is never undefined, we set

$f'(x) = 0$ so

$$3x^2 - 1 = 0$$

$$x^2 = \frac{1}{3} \rightarrow x = \pm\frac{1}{\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$



(SIGN PATTERN)

First we consider $x < -\frac{\sqrt{3}}{3}$, we try $x = -1$:

$$f'(-1) = 3(-1)^2 - 1 = 2 > 0$$

Next we consider $-\frac{\sqrt{3}}{3} < x < \frac{\sqrt{3}}{3}$, we try $x = 0$:

$$f'(0) = 3 \cdot 0^2 - 1 = -1 < 0$$

Lastly we consider $x > \frac{\sqrt{3}}{3}$, we try $x = 1$:

$$f'(1) = 3 \cdot 1^2 - 1 = 2 > 0$$

Thus $f(x)$ is increasing for $x < -\frac{\sqrt{3}}{3}$ and for $x > \frac{\sqrt{3}}{3}$. It is decreasing for $-\frac{\sqrt{3}}{3} < x < \frac{\sqrt{3}}{3}$.

Alternatively we could write these as $(-\infty, -\frac{\sqrt{3}}{3})$ and $(\frac{\sqrt{3}}{3}, \infty)$ for $f(x)$ increasing, and $(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ for decreasing.

Theorem : The First Derivative Test

If $x=p$ is a critical point of a function $f(x)$ then

① if $f'(x)>0$ to the left of $x=p$ and $f'(x)<0$ to the right of $x=p$, then $f(x)$ has a relative maximum at $x=p$

② if $f'(x)<0$ to the left of $x=p$ and $f'(x)>0$ to the right of $x=p$, then $f(x)$ has a relative minimum at $x=p$

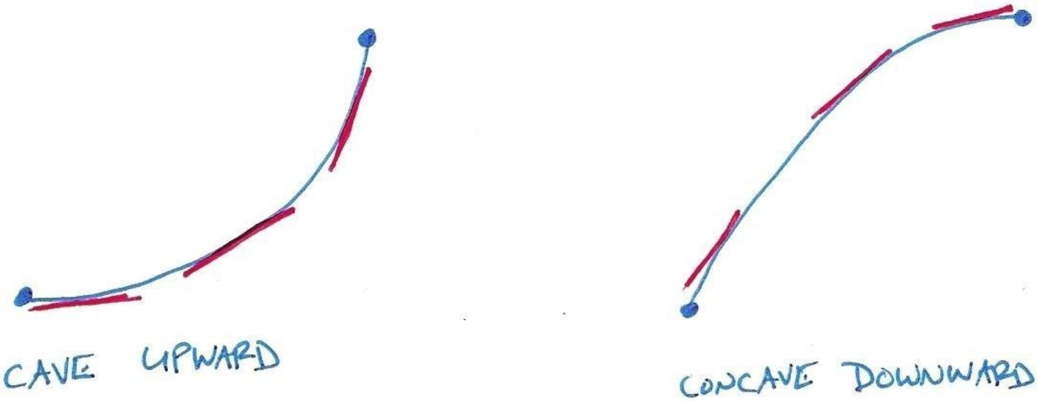
Otherwise, $f(x)$ has a saddle point at $x=p$, and therefore has no relative extremum there.

eg For $f(x)=x^3-x$, the critical point

$x=-\frac{\sqrt{3}}{3}$ is a relative maximum, while

$x=\frac{\sqrt{3}}{3}$ is a relative minimum.

A function that is increasing (or decreasing) can "bend" in different ways as it does so. We call this <u>concavity</u>. There are two basic types:



<u>Theorem</u>: Given a function $f(x)$ defined on an interval I,

① if $f''(x) > 0$ for all x in I then $f(x)$ is concave upward on I

② if $f''(x) < 0$ for all x in I then $f(x)$ is concave downward on I

<u>Def'n</u>: A <u>hypercritical point</u> of a function $f(x)$ is a point $x = p$ in the domain of $f(x)$ for which either $f''(p) = 0$ or $f''(p)$ is undefined.

Def'n: If $x=p$ is a hypercritical point of a function $f(x)$ at which the concavity of $f(x)$ changes, then we say $x=p$ is a <u>point of inflection</u>.

eg Find all the intervals on which $f(x)$ is increasing or decreasing, concave upward or concave downward, and classify all critical and hypercritical points given

$$f(x) = 3x^2 - x^3.$$

$$f'(x) = 6x - 3x^2$$

$$f''(x) = 6 - 6x$$

Since neither $f'(x)$ nor $f''(x)$ is undefined, we obtain critical and hypercritical points only if these are equal to zero.

First we set $6x - 3x^2 = 0$

$$3x(2-x) = 0 \rightarrow x=0 \quad x=2$$

(CRITICAL PTS)

$f'(x)$	$-$		$+$		$-$ → x
$f(x)$	↘	0	↗	2	↘

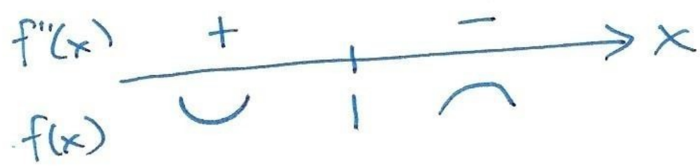
We see that $f(x)$ is increasing for $0 < x < 2$

decreasing for $x < 0$, $x > 2$

Thus $x = 0$ is a relative minimum, while $x = 2$ is a relative maximum.

Next we set $6 - 6x = 0$

$x = 1$ (HYPERCRITICAL POINT)



Here, $f(x)$ is concave upward for $x < 1$

concave downward for $x > 1$

Finally, we see that $x = 1$ is a point of inflection.