Section 3.7: Higher Derivatives

When we differentiate a function two or more times, we obtain <u>higher derivatives</u>. The derivative of the derivative is the <u>second derivative</u>. Its derivative is the <u>third derivative</u>. In general, if we differentiate n times, we obtain the <u>nth derivative</u>.

We often refer to the derivative of a function as its <u>first derivative</u>.

When we differentiate n times, we refer to n as the <u>order</u> of the derivative.

NOTATION FOR HIGHER DERIVATIVES

First derivative	y'	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}[f(x)]$
Second derivative	y''	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d^2}{dx^2}[f(x)]$
Third derivative	y'''	$f'''(x)$	$\frac{d^3y}{dx^3}$	$\frac{d^3}{dx^3}[f(x)]$
Fourth derivative	$y^{(4)}$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$	$\frac{d^4}{dx^4}[f(x)]$
nth derivative	$y^{(n)}$	$f^{(n)}(x)$	$\frac{d^ny}{dx^n}$	$\frac{d^n}{dx^n}[f(x)]$

In order to find the second derivative of a function, we start by obtaining its first derivative and then differentiate again. We could extend this approach to find any higher derivative.

eg Find the second derivative of $y = \sec(x)$.

$$\frac{dy}{dx} = \sec(x)\tan(x)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\sec(x)\tan(x)\right]$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\sec(x)\right]\tan(x) + \sec(x)\cdot\frac{d}{dx}\left[\tan(x)\right]$$

$$= \sec(x)\tan(x)\cdot\tan(x) + \sec(x)\cdot\sec^2(x)$$

$$\boxed{= \sec(x)\tan^2(x) + \sec^3(x)}$$

In some cases, we can observe patterns in the higher derivatives which we can use to help calculate them.

eg Find the 16^{th} derivative of $f(x) = 5x^3$.

$$f'(x) = 15x^2 \qquad f^{(4)}(x) = 0$$

$$f''(x) = 30x \qquad \vdots$$

$$f'''(x) = 30 \qquad \boxed{f^{(16)}(x) = 0}$$

eg Find the 100^{th} derivative of $y = \sin(x)$.

$$y' = \cos(x) \qquad y^{(5)} = \cos(x)$$

$$y'' = -\sin(x) \qquad y^{(6)} = -\sin(x)$$

$$y''' = -\cos(x) \qquad y^{(7)} = -\cos(x)$$

$$y^{(4)} = \sin(x) \qquad y^{(8)} = \sin(x)$$

Every fourth derivative is $\sin(x)$ and since $100 = 4 \cdot 25$ so $\boxed{y^{(100)} = \sin(x).}$

We can also find higher derivatives of implicit functions.

Whenever we calculate higher derivatives of an implicit function, we must always substitute our expression for the first derivative wherever $\frac{dy}{dx}$ appears. We often have the ability to simplify the higher derivatives of an implicit function by using the equation that defines it.

eg Find $\frac{d^2y}{dx^2}$ given $x^2 - y^2 = 7$.

We use implicit differentiation to find $\frac{dy}{dx}$:

$$\frac{d}{dx}\left[x^2 - y^2\right] = \frac{d}{dx}[7]$$

$$2x - 2y \cdot \frac{dy}{dx} = 0$$

$$-2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{x}{y}$$

We differentiate again:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{x}{y}\right] = \frac{\frac{d}{dx}[x] \cdot y - x \cdot \frac{d}{dx}[y]}{y^2}$$

$$= \frac{1 \cdot y - x \cdot \frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{y - x\cdot\left(\frac{x}{y}\right)}{y^2}$$

$$= \frac{y - \frac{x^2}{y}}{y^2}$$

$$= \frac{y^2 - x^2}{y^3}$$

$$\boxed{= \frac{-7}{y^3}}$$

Now recall that, for this implicit function,

$$x^2 - y^2 = 7$$

$$y^2 - x^2 = -7$$

In kinematics, acceleration is the rate of change of the velocity. Thus, if acceleration is denoted by $a(t)$ then

$$a(t) = v'(t) = s''(t).$$

eg A toy car moves in a straight line with position

$$s(t) = 2t^3 + 11t^2 - 8t + 1$$

where t is measured in seconds and position in centimetres. Find its initial acceleration.

$$v(t) = s'(t) = 6t^2 + 22t - 8$$

$$a(t) = s''(t) = 12t + 22$$

$$a(0) = 12\cdot 0 + 22 = 22$$

The initial acceleration is $\underline{22 \text{ cm/sec}^2}$.