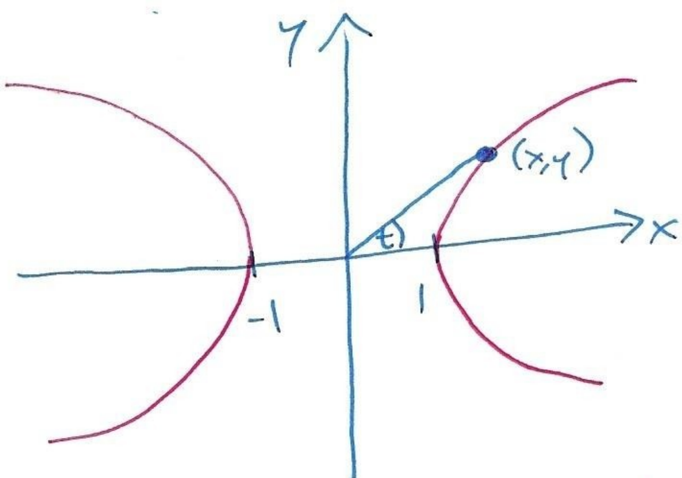
Section 3.6: Hyperbolic Functions and their Derivatives

Recall that the trigonometric functions can be defined using the unit circle $x^2+y^2=1$. We define $\cos(t)=x$ and $\sin(t)=y$ where t is the angle (or arclength). This is why

$$\cos^2(t)+\sin^2(t)=1.$$

In a similar way, we can define the <u>unit hyperbola</u> with equation $x^2-y^2=1$.



We define the x-coordinate of a point on the unit hyperbola to be the <u>hyperbolic cosine</u>, $\cosh(t)$.

We define the y-coordinate to be the <u>hyperbolic sine</u>, $\sinh(t)$.

Observe that $\cosh^2(t)-\sinh^2(t)=1$.

The other 4 hyperbolic functions are :

① hyperbolic tangent, $\tanh(t) = \dfrac{\sinh(t)}{\cosh(t)}$

② hyperbolic cotangent, $\coth(t) = \dfrac{\cosh(t)}{\sinh(t)} = \dfrac{1}{\tanh(t)}$

③ hyperbolic secant, $\operatorname{sech}(t) = \dfrac{1}{\cosh(t)}$

④ hyperbolic cosecant, $\operatorname{csch}(t) = \dfrac{1}{\sinh(t)}$

We can define

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \sinh(t) = \frac{e^t - e^{-t}}{2}$$

To see this, observe that

$$\left[\frac{e^t + e^{-t}}{2}\right]^2 - \left[\frac{e^t - e^{-t}}{2}\right]^2$$

$$= \frac{(e^t + e^{-t})(e^t + e^{-t})}{4} - \frac{(e^t - e^{-t})(e^t - e^{-t})}{4}$$

$$= \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4}$$

$$= \frac{4}{4} = 1$$

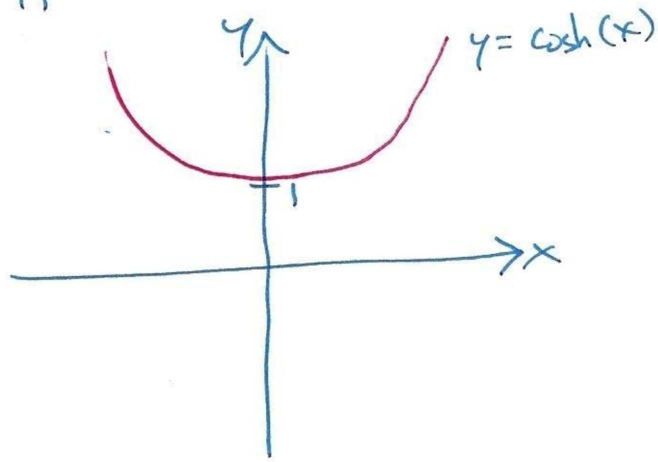
Thus they satisfy $x^2 - y^2 = 1$.

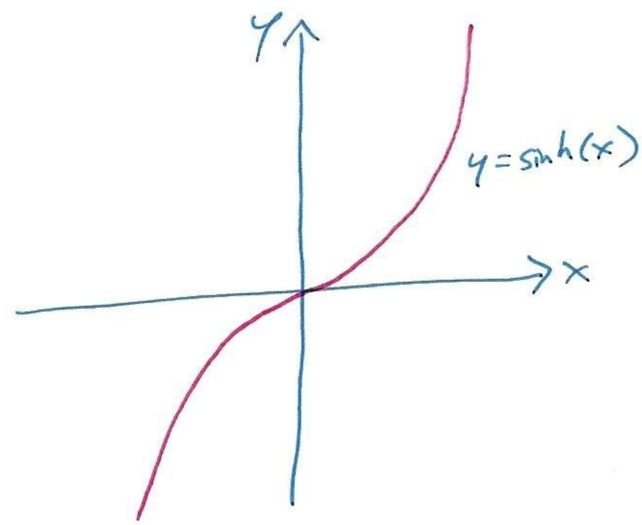
We can use this relationship to evaluate the hyperbolic functions:

$$\cosh(0) = \frac{e^0 + e^0}{2} = \frac{2}{2} = 1$$

$$\sinh(0) = \frac{e^0 - e^0}{2} = \frac{0}{2} = 0$$

It also allows us to visualize the graphs of the hyperbolic functions:





<u>Theorem</u>: ① $[\sinh(x)]' = \cosh(x)$

② $[\cosh(x)]' = \sinh(x)$

<u>Proof</u>: We will show that $[\cosh(x)]' = \sinh(x)$.

We write

$$\cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$\begin{aligned}
[\cosh(x)]' &= \left[\frac{e^{x}+e^{-x}}{2}\right]' \\
&= \frac{1}{2}\left([e^{x}]' + [e^{-x}]'\right) \\
&= \frac{1}{2}\left(e^{x} + e^{-x} \cdot [-x]'\right) \\
&= \frac{1}{2}\left(e^{x} - e^{-x}\right) \\
&= \frac{e^{x} - e^{-x}}{2} = \sinh(x)
\end{aligned}$$

eg $y = \cosh(x)\ln(x)$

$$\begin{aligned}
y' &= [\cosh(x)]'\ln(x) + \cosh(x) \cdot [\ln(x)]' \\
&= \sinh(x)\ln(x) + \cosh(x) \cdot \frac{1}{x} \\
&\boxed{= \sinh(x)\ln(x) + \frac{\cosh(x)}{x}}
\end{aligned}$$