# Section 3.5: Inverse Trigonometric Functions and their Derivatives

Given a function $f(x)$, a function $g(x)$ is said to be its <u>inverse</u> if:

① $f(x) = y$ implies that $g(y) = x$, and

② $g(x) = y$ implies that $f(y) = x$.

Note that, if $g(x)$ is the inverse of $f(x)$ then $f(x)$ is the inverse of $g(x)$.

Graphically, if $(p,q)$ is a point on the graph of $y = f(x)$ then $(q,p)$ is a point on the graph of $y = g(x)$.

This also means that the domain of $f(x)$ is the range of $g(x)$, and the range of $f(x)$ is the domain of $g(x)$.

The inverse of $f(x)$ is usually denoted by $f^{-1}(x)$. Note that

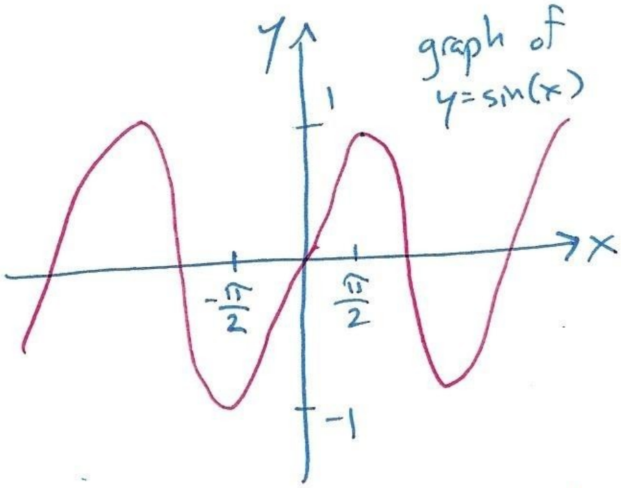
$$f^{-1}(x) \neq [f(x)]^{-1}$$

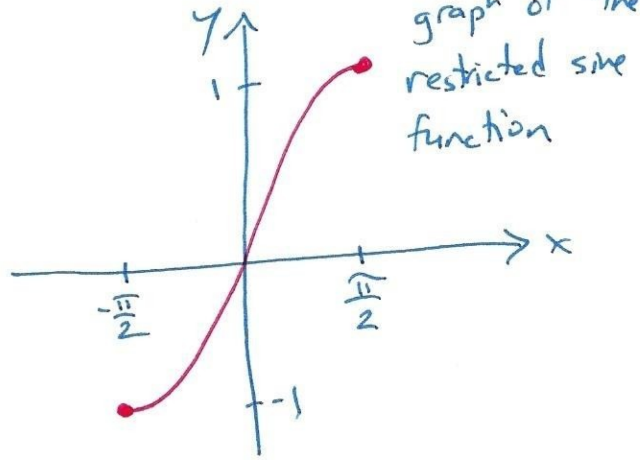
A function $f(x)$ and its inverse $f^{-1}(x)$ always obey cancellation properties on the appropriate domains:

$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.$$

Not all functions are <u>invertible</u> (have an inverse). Only functions for which we can draw any horizontal line through its graph and intersect the graph at most once are invertible. This is the <u>Horizontal Line Test</u>.

Consider the sine function, $y=\sin(x)$, which like all trigonometric functions, has a graph that fails the Horizontal Line Test:



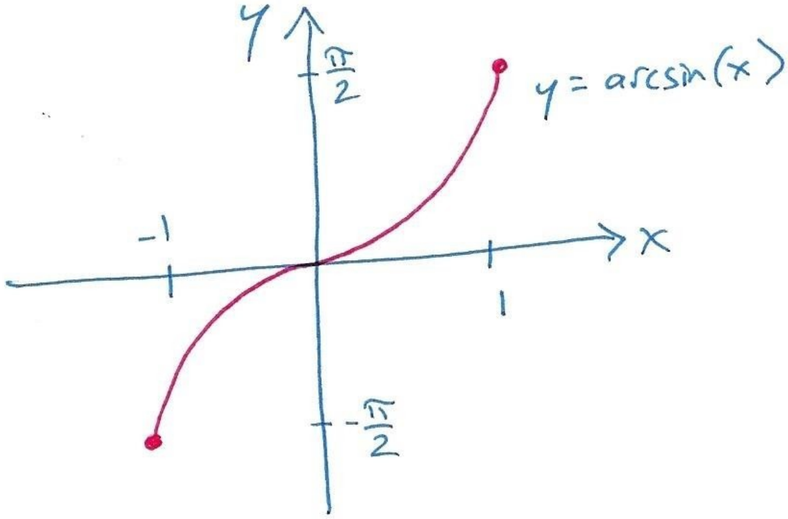

We define the restricted sine function to be equal to $\sin(x)$ but only for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.

The restricted sine function must have an inverse.

This inverse sine function is called the arcsine function. It is denoted by $\sin^{-1}(x)$ or $\arcsin(x)$.

The domain of $\arcsin(x)$ is $-1 \leq x \leq 1$.

The range of $\arcsin(x)$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.




To evaluate the arcsine function, where possible we apply our knowledge of the special angles in reverse.

eg $\arcsin\left(\frac{1}{2}\right)$

We need to determine a number $\theta$ for which $\sin(\theta) = \frac{1}{2}$ with $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$.

We know that $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ so

$$\boxed{\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}.}$$

The cancellation laws are:

$$\arcsin(\sin(x)) = x \quad \text{for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

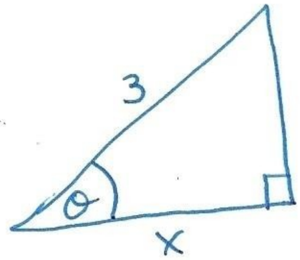
$$\sin(\arcsin(x)) = x \quad \text{for } -1 \le x \le 1$$

We can also combine arcsine with other trig functions.

eg $\tan(\arcsin(1/3))$

Let $\theta = \arcsin(1/3)$ so $\sin(\theta) = \frac{1}{3}$.

Then we can draw a right triangle with $\theta$ as an interior angle, opposite sidelength 1, and hypotenuse 3.




By the Pythagorean theorem, the adjacent side length is

$$x^2 + 1^2 = 3^2$$

$$x^2 = 8 \rightarrow x = \sqrt{8}$$

$$= 2\sqrt{2}$$

Then $\tan\left(\arcsin\left(\frac{1}{3}\right)\right)$

$$= \tan(\theta)$$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \boxed{= \frac{\sqrt{2}}{4}}$$

<u>Theorem</u>: $\left[\arcsin(x)\right]' = \dfrac{1}{\sqrt{1-x^2}}$

<u>Proof</u>: Let $y = \arcsin(x)$ so $x = \sin(y)$.

We differentiate implicitly:

$$\frac{d}{dx}[x] = \frac{d}{dx}[\sin(y)]$$

$$1 = \cos(y) \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$

Now recall that

$$\sin^2(y) + \cos^2(y) = 1$$

$$\cos^2(y) = 1 - \sin^2(y)$$

$$\cos(y) = \pm\sqrt{1 - \sin^2(y)}$$

Because $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, it must be that $\cos(y) \geq 0$.

Then $\cos(y) = \sqrt{1-\sin^2(y)}$ so

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}.$$

eg $f(x) = \arcsin(x^2-1)$

$$f'(x) = \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot [x^2-1]'$$

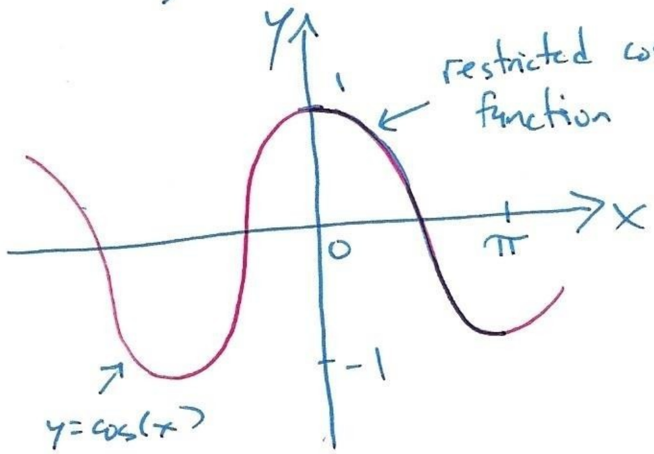
$$= \frac{1}{\sqrt{1-(x^2-1)^2}} \cdot 2x$$

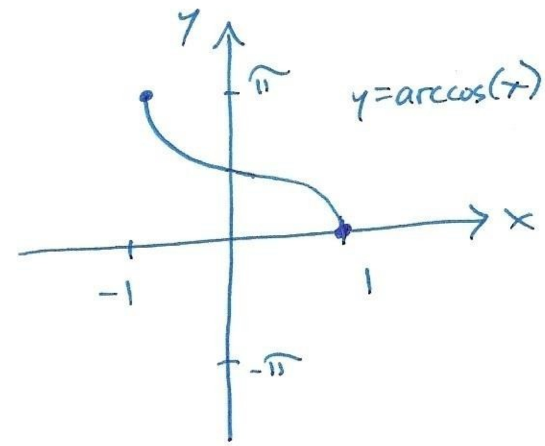
$$= \frac{2x}{\sqrt{1-(x^4-2x^2+1)}}$$

$$\boxed{= \frac{2x}{\sqrt{2x^2-x^4}}}$$

The inverse cosine function is called the <u>arccosine</u> function, and is denoted by $\arccos(x)$.

It is the inverse of a <u>restricted</u> cosine function, defined on the domain $0 \le x \le \pi$.




The domain of $\arccos(x)$ is $-1 \le x \le 1$, and its range $0 \le y \le \pi$.

The cancellation laws are

$$\arccos(\cos(x)) = x \quad \text{for } 0 \le x \le \pi$$

$$\cos(\arccos(x)) = x \quad \text{for } -1 \le x \le 1$$

eg $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

Theorem: $\left[\arccos(x)\right]' = \dfrac{-1}{\sqrt{1-x^2}}$

eg $f(x) = \sin(\arccos(x))$

$$f'(x) = \cos(\arccos(x)) \cdot \left[\arccos(x)\right]'$$

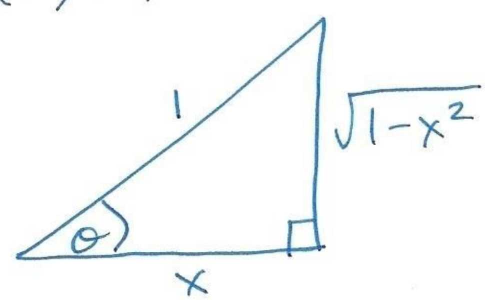
$$= \cos(\arccos(x)) \cdot \left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$= x \cdot \left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\boxed{= \frac{-x}{\sqrt{1-x^2}}}$$

Alternatively, we could let $\theta = \arccos(x)$ so

$\cos(\theta) = x$




Then

$$f(x) = \sin(\theta)$$

$$= \frac{\sqrt{1-x^2}}{1}$$

$$= \sqrt{1-x^2}.$$

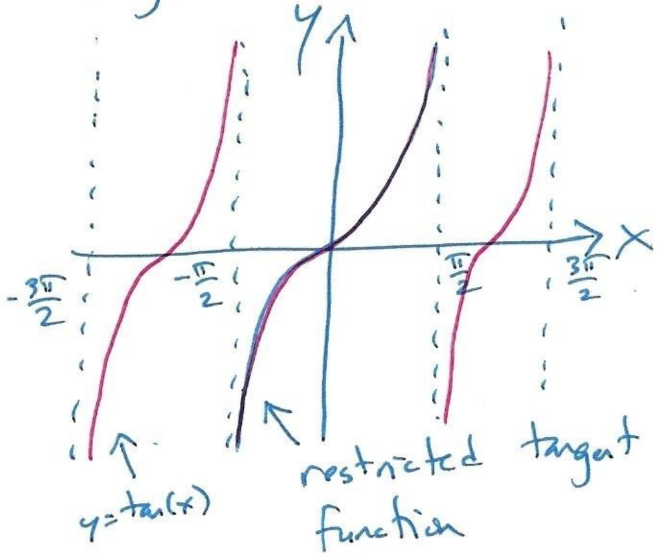
Now $f'(x) = \frac{1}{2}(1-x^2)^{-1/2} \cdot \left[1-x^2\right]'$

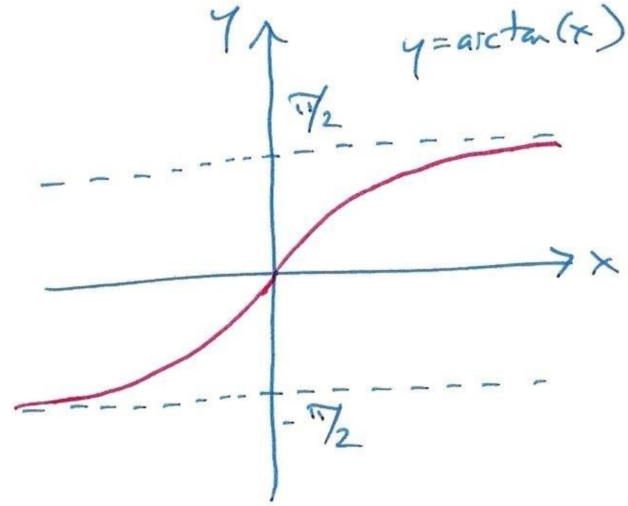
$$= \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)$$

$$\boxed{= \frac{-x}{\sqrt{1-x^2}}}$$

The inverse tangent function is called the <u>arctangent</u> function, denoted by $\arctan(x)$.

In this case, we restrict the domain of the tangent function to $-\frac{\pi}{2} < x < \frac{\pi}{2}$.




The domain of $\arctan(x)$ is $-\infty < x < \infty$ or $\mathbb{R}$ (the set of all real numbers). Its range is $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

eg $y = x\arctan(\sqrt{x})$

$$y' = [x]'\arctan(\sqrt{x}) + x\,[\arctan(\sqrt{x})]'$$

$$= 1\cdot\arctan(\sqrt{x}) + x\cdot\frac{1}{(\sqrt{x})^2+1}\cdot[\sqrt{x}]'$$

$$= \arctan(\sqrt{x}) + x\cdot\frac{1}{x+1}\cdot\frac{1}{2}x^{-1/2}$$

$$\boxed{= \arctan(\sqrt{x}) + \frac{\sqrt{x}}{2(x+1)}}$$

The arccotangent function, arccot(x), is defined similarly to arctan(x) except that is range is $0 < y < \pi$.

<u>Theorem</u>: $[\text{arccot}(x)]' = \dfrac{-1}{x^2+1}$

For the arcsec(x) and arccsc(x) functions we have the identities

$$\text{arcsec}(x) = \arccos\left(\frac{1}{x}\right)$$

$$\text{arccsc}(x) = \arcsin\left(\frac{1}{x}\right)$$

eg $\text{arccsc}(2) = \arcsin\left(\frac{1}{2}\right) \boxed{= \frac{\pi}{6}}$

<u>Theorem</u>: $[\text{arcsec}(x)]' = \dfrac{1}{x\sqrt{x^2-1}}$

$$[\text{arccsc}(x)]' = \frac{-1}{x\sqrt{x^2-1}}$$

eg $y = \text{arcsec}(x^4)$

$$\frac{dy}{dx} = \frac{1}{x^4\sqrt{(x^4)^2-1}} \cdot \frac{d}{dx}[x^4]$$

$$= \frac{1}{x^4\sqrt{x^8-1}} \cdot 4x^3$$

$$\boxed{= \frac{4}{x\sqrt{x^8-1}}}$$