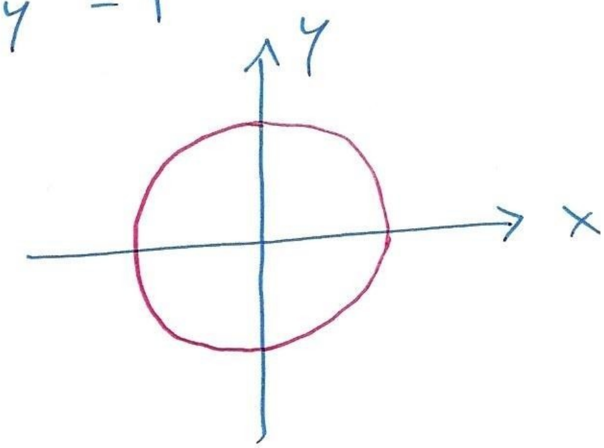
Section 3.3: Implicit Differentiation

Recall that a <u>function</u> is a rule which associates a <u>unique</u> value y in the <u>range</u> to each value of x in the <u>domain</u>. Consequently, we must be able to draw a vertical line through any part of the graph of a function $y=f(x)$ and have it intersect that graph at most <u>once</u>. This is the Vertical Line Test.

Consequently, there are many graphs which cannot be the graph of a function because they fail the Vertical Line Test.

eg $x^2+y^2=1$



An equation which establishes a relationship between the variables x and y is called an implicit function.

These are sometimes equivalent to (explicit) functions.

eg $y - 3x^2 + 5 = 0$

This can be rewritten as

$$y = 3x^2 - 5$$

Most often, as with $x^2 + y^2 = 1$, they cannot but we still want to quantify the derivative $\frac{dy}{dx}$. Finding it is called implicit differentiation.

Given an implicit function, typically an equation involving the variables x and y, we want to find the derivative $\frac{dy}{dx}$. How do we take the derivative with respect to x of an expression involving y?

Suppose that $y = f(x)$.

eg $\frac{d}{dx}\left[y^4\right]$

Compare : $\frac{d}{dx}\left[(f(x))^4\right] = 4(f(x))^3 \cdot \frac{d}{dx}\left[f(x)\right]$

Thus : $\frac{d}{dx}\left[y^4\right] = 4y^3 \cdot \frac{dy}{dx}$ by the Chain Rule

eg $\frac{d}{dx}\left[\sin(y)\right]$

Compare : $\frac{d}{dx}\left[\sin(f(x))\right] = \cos(f(x)) \cdot \frac{d}{dx}\left[f(x)\right]$

Thus : $\frac{d}{dx}\left[\sin(y)\right] = \cos(y) \frac{dy}{dx}$

eg $\frac{d}{dx}\left[x^2\sqrt{y}\right]$

$$= \frac{d}{dx}\left[x^2\right]\sqrt{y} + x^2 \cdot \frac{d}{dx}\left[\sqrt{y}\right]$$

$$= 2x\sqrt{y} + x^2 \cdot \frac{1}{2} y^{-1/2} \cdot \frac{dy}{dx}$$

$$\boxed{= 2x\sqrt{y} + \frac{x^2}{2\sqrt{y}} \cdot \frac{dy}{dx}}$$

How do we apply this approach to an equation involving x and y?

eg Find $\frac{dy}{dx}$ given $x^2 + y^2 = 1$.

One approach is to try to solve for y in terms of x:

$$y^2 = 1 - x^2$$

$$y = \pm\sqrt{1-x^2}$$

Thus the unit circle is a combination of two functions: $y = \sqrt{1-x^2}$ (upper semi-circle) and $y = -\sqrt{1-x^2}$ (lower semi-circle).

For $y = \sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{1}{2}(1-x^2)^{-1/2} \cdot \frac{d}{dx}\left[1-x^2\right]$$

$$= \frac{1}{2}(1-x^2)^{-1/2} \cdot (-2x)$$

$$= -\frac{x}{\sqrt{1-x^2}} = -\frac{x}{y}$$

For $y = -\sqrt{1-x^2}$

$$\frac{dy}{dx} = -\left(-\frac{x}{\sqrt{1-x^2}}\right) = \frac{x}{\sqrt{1-x^2}} = \frac{x}{-y} = -\frac{x}{y}$$

Alternatively, we could differentiate both sides of the equation with respect to x:

$$\frac{d}{dx}\left[x^2+y^2\right] = \frac{d}{dx}\left[1\right]$$

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = 0$$

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$2y \cdot \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

For implicit differentiation, we start by differentiating both sides of the equation. Then we solve for the derivative.

eg Find $\frac{dy}{dx}$ given $e^{x^2} + e^{y^2} = x - y$.

$$\frac{d}{dx}\left[e^{x^2} + e^{y^2}\right] = \frac{d}{dx}\left[x - y\right]$$

$$\frac{d}{dx}\left[e^{x^2}\right] + \frac{d}{dx}\left[e^{y^2}\right] = \frac{d}{dx}\left[x\right] - \frac{d}{dx}\left[y\right]$$

$$e^{x^2} \cdot 2x + e^{y^2} \cdot 2y \cdot \frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$2ye^{y^2}\frac{dy}{dx} + \frac{dy}{dx} = 1 - 2xe^{x^2}$$

$$\frac{dy}{dx}\left(2ye^{y^2} + 1\right) = 1 - 2xe^{x^2}$$

$$\boxed{\frac{dy}{dx} = \frac{1 - 2xe^{x^2}}{2ye^{y^2} + 1}}$$

eg Find the equations of the tangent and normal lines to the curve defined by

$$\sin(xy) = 1 - \cos(x) - \cos(y)$$

at the point $(0, \pi/2)$.

We use implicit differentiation:

$$\frac{d}{dx}\left[\sin(xy)\right] = \frac{d}{dx}\left[1 - \cos(x) - \cos(y)\right]$$

$$\cos(xy) \cdot \frac{d}{dx}\left[xy\right] = 0 - \left[-\sin(x)\right] - \left[-\sin(y)\right] \cdot \frac{dy}{dx}$$

$$\cos(xy) \cdot \left(\frac{d}{dx}[x]y + x\frac{d}{dx}[y]\right) = \sin(x) + \sin(y) \cdot \frac{dy}{dx}$$

$$\cos(xy) \cdot \left(y + x\frac{dy}{dx}\right) = \sin(x) + \sin(y) \cdot \frac{dy}{dx}$$

$$x\cos(xy) \cdot \frac{dy}{dx} - \sin(y) \cdot \frac{dy}{dx} = \sin(x) - y\cos(xy)$$

$$\frac{dy}{dx}\left[x\cos(xy) - \sin(y)\right] = \sin(x) - y\cos(xy)$$

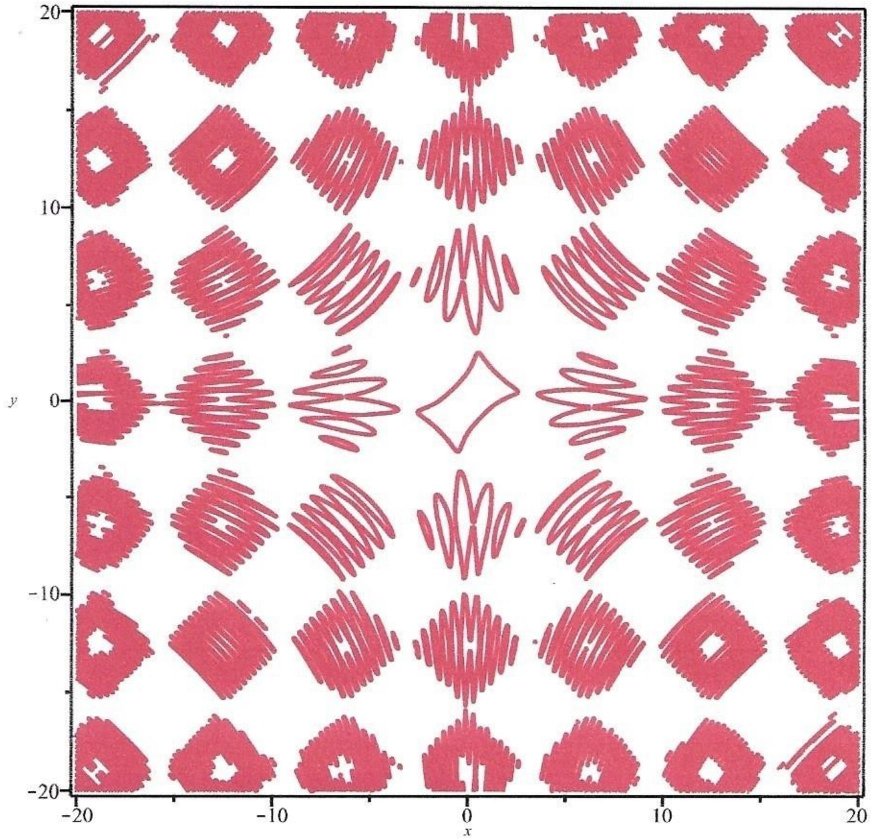
$$\frac{dy}{dx} = \frac{\sin(x) - y\cos(xy)}{x\cos(xy) - \sin(y)}$$

The slope of the tangent line at $(0, \pi/2)$ is

$$m_T = \frac{dy}{dx} = \frac{\sin(0) - \frac{\pi}{2} \cdot \cos(0)}{0 - \sin(\pi/2)} = \frac{-\pi/2}{-1} = \frac{\pi}{2}$$

Thus the slope of the normal line is

$$m_N = -\frac{1}{m_T} = -\frac{2}{\pi}$$



Graph of

$\sin(xy) = 1 - \cos(x) - \cos(y)$

The equation of the tangent line is

$$y - \frac{\pi}{2} = \frac{\pi}{2}(x - 0)$$

$$\boxed{y = \frac{\pi}{2}x + \frac{\pi}{2}}$$

The equation of the normal line is

$$y - \frac{\pi}{2} = -\frac{2}{\pi}(x-0)$$

$$\boxed{y = -\frac{2}{\pi}x + \frac{\pi}{2}}$$