Section 3.2: The Chain Rule

Some <u>composite</u> functions can be differentiated by rewriting them as a sum or difference of <u>simple</u> functions.

eg $y = (x^3 + 5x)^2$

This is a composite function, but it can be rewritten as

$$y = x^6 + 10x^4 + 25x^2$$

$$\boxed{\frac{dy}{dx} = 6x^5 + 40x^3 + 50x}$$

However, this is not always feasible or even possible.

eg $y = (x^3 + 5x)^{200}$ or $y = \cot(x^3 + 5x)$

When we divide a composite function $y = f(g(x))$ into the constituent simple functions

$$y = f(u) \quad \text{and} \quad u = g(x)$$

this is called <u>decomposing</u> the composite function.

eg $y = (x^3+5x)^2$

We can write this function as $y = f(g(x))$

where $f(x) = x^2$

$g(x) = x^3 + 5x$

or as $y = u^2$ where $u = x^3 + 5x$.

eg $y = \sin^4(x) = [\sin(x)]^4$

$f(x) = x^4$ and $g(x) = \sin(x)$

so $y = u^4$ where $u = \sin(x)$

Some composite functions can only be written as the composition of 3 or more simple functions.

eg $y = \sin(e^{5x})$

$f(x) = \sin(x)$, $g(x) = e^x$, $h(x) = 5x$

so $y = \sin(u)$ where $u = e^v$

and $v = 5x$

Our hope is that there will be a connection between the derivative of a composite functions and the derivatives of the simple functions which constitute it.

eg $y = (x^3 + 5x)^2$

$$\frac{dy}{dx} = 6x^5 + 40x^3 + 50x$$

$y = u^2$ where $u = x^3 + 5x$

For $y = u^2$: $\frac{dy}{du} = 2u = 2(x^3+5x)$

For $u = x^3+5x$: $\frac{du}{dx} = 3x^2 + 5$

Then $\frac{dy}{du} \cdot \frac{du}{dx} = 2(x^3+5x) \cdot (3x^2+5)$

$$= 2(3x^5 + 5x^3 + 15x^3 + 25x)$$

$$= 6x^5 + 40x^3 + 50x$$

$$= \frac{dy}{dx}$$

Theorem: The Chain Rule

Given a composite function $y = f(x)$ which can be decomposed into $y = f(u)$ and $u = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

or, equivalently, $y' = f'(g(x))\, g'(x)$.

eg $y = \sin(e^x)$

$f(x) = \sin(x)$ and $g(x) = e^x$

so $y = \sin(u)$ where $u = e^x$

$\frac{dy}{du} = \cos(u) = \cos(e^x)$ $\qquad \frac{du}{dx} = e^x$

Thus $\frac{dy}{dx} = \cos(e^x) \cdot e^x$

$$\boxed{= e^x \cos(e^x)}$$

eg $y = \frac{1}{\sqrt[4]{1-x^2}} = (1-x^2)^{-1/4}$

$$y' = -\frac{1}{4}(1-x^2)^{-5/4} \cdot [1-x^2]'$$

$$= -\frac{1}{4}(1-x^2)^{-5/4} \cdot (-2x)$$

$$\boxed{= \frac{1}{2} x (1-x^2)^{-5/4}}$$

eg $f(x) = \sec(2x^3)$

$$f'(x) = \sec(2x^3)\tan(2x^3) \cdot [2x^3]'$$

$$= \sec(2x^3)\tan(2x^3) \cdot 6x^2$$

$$\boxed{= 6x^2 \sec(2x^3)\tan(2x^3)}$$

We often need to combine the Chain Rule with the Product Rule.

eg $f(x) = (x^2+x)^5(x^3+1)^2$

Here we apply the Product Rule first, then the Chain Rule (twice):

$$f'(x) = [(x^2+x)^5]'(x^3+1)^2 + (x^2+x)^5[(x^3+1)^2]'$$

Next,

$$[(x^2+x)^5]' = 5(x^2+x)^4 \cdot [x^2+x]'$$

$$= 5(x^2+x)^4 \cdot (2x+1)$$

$$[(x^3+1)^2]' = 2(x^3+1) \cdot [x^3+1]'$$

$$= 2(x^3+1) \cdot 3x^2$$

$$= 6x^2(x^3+1)$$

Hence $f'(x) = 5(x^2+x)^4(2x+1) \cdot (x^3+1)^2$
$+ (x^2+x)^5 \cdot 6x^2(x^3+1)$

$$f'(x) = \boxed{5(x^2+x)^4(2x+1)(x^3+1)^2 + 6x^2(x^2+x)^5(x^3+1)}$$

$$= (x^2+x)^4(x^3+1)\left[5(2x+1)(x^3+1) + 6x^2(x^2+x)\right]$$

$$= (x^2+x)^4(x^3+1)\left[5(2x^4+x^3+2x+1) + 6x^4+6x^3\right]$$

$$\boxed{= (x^2+x)^4(x^3+1)(16x^4+11x^3+10x+5)}$$

eg $y = \cos(x^2e^x)$

First we apply the Chain Rule, then the Product Rule:

$$\frac{dy}{dx} = -\sin(x^2e^x)\cdot\frac{d}{dx}\left[x^2e^x\right]$$

$$= -\sin(x^2e^x)\cdot\left(\frac{d}{dx}\left[x^2\right]e^x + x^2\cdot\frac{d}{dx}\left[e^x\right]\right)$$

$$= -\sin(x^2e^x)\cdot\left(2xe^x + x^2e^x\right)$$

$$= -xe^x\sin(x^2e^x)\cdot(2+x)$$

$$\boxed{= -x(x+2)e^x\sin(x^2e^x)}$$

Likewise, we can combine the Chain Rule with the Quotient Rule.

eg $y = \dfrac{x}{\sqrt{1+x^4}}$

We apply the Quotient Rule, then the Chain Rule.

$$y' = \frac{[x]'\sqrt{1+x^4} - x\left[\sqrt{1+x^4}\right]'}{\left(\sqrt{1+x^4}\right)^2}$$

$$= \frac{1 \cdot \sqrt{1+x^4} - x \cdot \frac{1}{2}(1+x^4)^{-1/2} \cdot [1+x^4]'}{1+x^4}$$

$$= \frac{\sqrt{1+x^4} - x \cdot \frac{1}{2}(1+x^4)^{-1/2} \cdot 4x^3}{1+x^4}$$

$$= \frac{\sqrt{1+x^4} - 2x^4(1+x^4)^{-1/2}}{1+x^4} \cdot \frac{(1+x^4)^{1/2}}{(1+x^4)^{1/2}}$$

$$= \frac{1+x^4-2x^4}{(1+x^4)^{3/2}} \quad \boxed{= \frac{1-x^4}{(1+x^4)^{3/2}}}$$

Now suppose we have a function $y = f(x)$ which can be decomposed into 3 simple functions:

$$y = f(u), \quad u = g(v), \quad v = h(x)$$

Then the Chain Rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \qquad \text{where } u = g(h(x))$$

We need to apply the Chain Rule again:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

eg $y = \sin^3(x^5) = [\sin(x^5)]^3$

$f(x) = x^3$, $g(x) = \sin(x)$, $h(x) = x^5$

$y = u^3$, $u = \sin(v)$, $v = x^5$

$\frac{dy}{du} = 3u^2$ $\quad$ $\frac{du}{dv} = \cos(v)$ $\quad$ $\frac{dv}{dx} = 5x^4$

$= 3\sin^2(v)$ $\quad$ $= \cos(x^5)$

$= 3\sin^2(x^5)$

By the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= 3\sin^2(x^5) \cdot \cos(x^5) \cdot 5x^4$$

$$\boxed{= 15x^4 \sin^2(x^5)\cos(x^5)}$$

eg $f(x) = e^{\cos(2x)}$

$$f'(x) = e^{\cos(2x)} \cdot [\cos(2x)]'$$

$$= e^{\cos(2x)} \cdot [-\sin(2x)] \cdot [2x]'$$

$$= e^{\cos(2x)} \cdot [-\sin(2x)] \cdot 2$$

$$\boxed{= -2e^{\cos(2x)}\sin(2x)}$$

eg $f(x) = \sqrt{\sin(x) + (1-x)^4}$

$$f'(x) = \frac{1}{2}\left[\sin(x) + (1-x)^4\right]^{-1/2} \cdot \left[\sin(x) + (1-x)^4\right]'$$

$$= \frac{1}{2\sqrt{\sin(x)+(1-x)^4}} \cdot \left[\cos(x) + \left[(1-x)^4\right]'\right]$$

$$= \frac{1}{2\sqrt{\sin(x)+(1-x)^4}} \cdot \left[\cos(x) + 4(1-x)^3 \cdot [1-x]'\right]$$

$$= \frac{1}{2\sqrt{\sin(x)+(1-x)^4}} \cdot \left[\cos(x) + 4(1-x)^3 \cdot (-1)\right]$$

$$\boxed{= \frac{\cos(x) - 4(1-x)^3}{2\sqrt{\sin(x)+(1-x)^4}}}$$

Theorem: For any positive number $b \neq 1$,

$$\frac{d}{dx}\left[b^x\right] = b^x \ln(b).$$

Note that, if $b = e$, this becomes

$$\frac{d}{dx}\left[e^x\right] = e^x \ln(e)$$

$$= e^x \log_e(e)$$

$$= e^x \cdot 1 = e^x$$

as before.

Proof: We rewrite $b^x = (e^{\ln(b)})^x = e^{x\ln(b)}$.

Now we apply the Chain Rule:

$$\frac{d}{dx}\left[b^x\right] = \frac{d}{dx}\left[e^{x\ln(b)}\right]$$

$$= e^{x\ln(b)} \cdot \frac{d}{dx}\left[x\ln(b)\right]$$

$$= e^{x\ln(b)} \cdot \ln(b)$$

$$= b^x \ln(b).$$

eg $y = 5^x$

$$\boxed{y' = 5^x \ln(5)}$$

eg $f(x) = 3^{\sec(x)}$

$$f'(x) = 3^{\sec(x)} \ln(3) \cdot \left[\sec(x)\right]'$$

$$\boxed{= 3^{\sec(x)} \ln(3) \sec(x) \tan(x)}$$