Section 2.4: The Product and Quotient Rules

Unfortunately,

$$[f(x)g(x)]' \neq f'(x)\,g'(x)$$

$$\left[\frac{f(x)}{g(x)}\right]' \neq \frac{f'(x)}{g'(x)}$$

To see this, observe the use of the Constant Multiple Rule:

$$[6x]' = 6\,[x]' = 6 \cdot 1 = 6$$

The following is <u>not</u> true:

$$[6x]' = [6]' \cdot [x]' = 0 \cdot 1 = 0$$

Theorem: The Product Rule

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

Proof: Let $A(x) = f(x)g(x)$ so

$$[f(x)g(x)]' = A'(x)$$

$$= \lim_{h\to 0} \frac{A(x+h) - A(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h)}{h}$$

$$+ \lim_{h\to 0} \frac{f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h)\right]$$

$$+ f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right] \cdot \left[\lim_{h\to 0} g(x+h)\right]$$

$$+ f(x) \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) \cdot g\left(\lim_{h \to 0} (x+h)\right) + f(x) \cdot g'(x)$$

$$= f'(x) g(x) + f(x) g'(x)$$

eg $y = (7x+1)(x^4 - x^3 - 9x)$

$$y' = [7x+1]'(x^4 - x^3 - 9x) + (7x+1)[x^4 - x^3 - 9x]'$$

$$= (7+0)(x^4 - x^3 - 9x) + (7x+1)(4x^3 - 3x^2 - 9)$$

$$\boxed{\begin{aligned} &= 7(x^4 - x^3 - 9x) + (7x+1)(4x^3 - 3x^2 - 9) \\ &= 35x^4 - 24x^3 - 3x^2 - 126x - 9 \end{aligned}}$$

eg $f(x) = 2x(x+1)(3x^2-1)$

To apply the Product Rule, we will treat $(x+1)(3x^2-1)$ as a single function:

$$f'(x) = [2x]'(x+1)(3x^2-1) + 2x\left[(x+1)(3x^2-1)\right]'$$
$$= 2(x+1)(3x^2-1) + 2x\left[(x+1)(3x^2-1)\right]'$$

Now we'll apply the Product Rule to the remaining derivative:

$$\left[(x+1)(3x^2-1)\right]' = [x+1]'(3x^2-1) + (x+1)[3x^2-1]'$$
$$= 1\cdot(3x^2-1) + (x+1)(6x-0)$$
$$= 3x^2-1+6x^2+6x$$
$$= 9x^2+6x-1$$

Now we have

$$\boxed{\begin{aligned} f'(x) &= 2(x+1)(3x^2-1) + 2x(9x^2+6x-1) \\ &= 24x^3+18x^2-4x-2 \end{aligned}}$$

Theorem: The Quotient Rule

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

eg $f(x) = \dfrac{4x+5}{2-3x^3}$

$$f'(x) = \frac{[4x+5]'(2-3x^3) - (4x+5)[2-3x^3]'}{(2-3x^3)^2}$$

$$= \frac{(4+0)(2-3x^3) - (4x+5)(0-9x^2)}{(2-3x^3)^2}$$

$$= \frac{8-12x^3 + 36x^3 + 45x^2}{(2-3x^3)^2}$$

$$\boxed{= \frac{24x^3 + 45x^2 + 8}{(2-3x^3)^2}}$$

eg $y = \dfrac{8+x^2}{4x^2}$

We could use the Quotient Rule:

$$\frac{dy}{dx} = \frac{\frac{d}{dx}[8+x^2]\cdot(4x^2) - (8+x^2)\cdot\frac{d}{dx}[4x^2]}{(4x^2)^2}$$

$$= \frac{2x\cdot(4x^2) - (8+x^2)\cdot 8x}{16x^4}$$

$$= \frac{8x^3 - 64x - 8x^3}{16x^4}$$

$$= \frac{-64x}{16x^4} = -\frac{4}{x^3} = -4x^{-3}$$

However, we could instead write

$$y = \frac{8}{4x^2} + \frac{x^2}{4x^2}$$

$$= 2x^{-2} + \frac{1}{4}$$

$$\frac{dy}{dx} = 2 \cdot (-2x^{-3}) + 0$$

$$\boxed{= -4x^{-3}}$$