Section 2.2: The Limit Definition of the Derivative

Def'n: The derivative of a function $f(x)$ with respect to x at a point $x=p$ is given by
$$f'(p) = \lim_{h\to 0} \frac{f(p+h)-f(p)}{h}.$$

We say that $f(x)$ is differentiable at $x=p$ if $f'(p)$ can be assigned a value, that is, if the limit exists. The process of finding the derivative is called differentiation.

Alternatively, we could set $x=p+h$ so $h=x-p$. Then, as $h\to 0$, $x\to p$. Hence we could instead write
$$f'(p) = \lim_{x\to p} \frac{f(x)-f(p)}{x-p}.$$

We call this the alternative def'n of the derivative.

eg Find the derivative of $f(x) = 5-x^2$ at the point $x=3$.

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h\to 0} \frac{[5-(3+h)^2] - [5-3^2]}{h}$$

$$= \lim_{h\to 0} \frac{5-9-6h-h^2+4}{h}$$

$$= \lim_{h\to 0} \frac{-6h-h^2}{h}$$

$$= \lim_{h\to 0} (-6-h) \boxed{= -6}$$

eg Find the derivative of $f(x) = 5-x^2$ at the points $x=0$, $x=1$ and $x=-1$.

Instead of repeating this calculation 3 more times, let's evaluate $f'(x)$ for some unspecified value $p=x$:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[5-(x+h)^2] - [5-x^2]}{h}$$

$$= \lim_{h\to 0} \frac{[5-x^2-2xh-h^2]-5+x^2}{h}$$

$$= \lim_{h\to 0} \frac{-2xh-h^2}{h}$$

$$= \lim_{h\to 0} (-2x - h)$$

$$\boxed{= -2x}$$

Thus $f'(0) = -2\cdot 0 \boxed{= 0}$

$f'(1) = -2\cdot 1 \boxed{= -2}$

$f'(-1) = -2\cdot(-1) \boxed{= 2}$

<u>Def'n</u>: The <u>derivative</u> of a function $f(x)$ with respect to x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

eg Differentiate $f(x) = x^3 - 7x$.

$$f'(x) = \lim_{h\to 0} \frac{[(x+h)^3 - 7(x+h)] - [x^3 - 7x]}{h}$$

$$= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 7x - 7h - x^3 + 7x}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3 - 7h}{h}$$

$$= \lim_{h\to 0} (3x^2 + 3xh + h^2 - 7)$$

$$\boxed{= 3x^2 - 7}$$

eg Find the derivative of $f(x) = 2x+3$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[2(x+h)+3] - [2x+3]}{h}$$

$$= \lim_{h\to 0} \frac{2x+2h+3-2x-3}{h}$$

$$= \lim_{h\to 0} \frac{2h}{h}$$

$$= \lim_{h\to 0} 2 \boxed{= 2}$$

Prime notation denotes the derivative of a function

$y = f(x)$ as $f'(x)$ or y'.

Leibniz notation instead writes it as

$$\frac{d}{dx}[f(x)] \text{ or } \frac{dy}{dx}.$$

How can $f(x)$ be non-differentiable at $x=p$?

1. The one-sided limits of the derivative could be infinite limits, resulting in a vertical tangent line at $x=p$.

   eg $f(x) = \sqrt[3]{x}$ at $x=0$

2. The one-sided limits of the derivative may not be equal, resulting in a "sharp corner" in the graph at $x=p$.

eg Consider $f(x) = |x-4|$ at $x=4$.

Recall that

$$|x-4| = \begin{cases} x-4, & \text{for } x \geq 4 \\ -(x-4), & \text{for } x < 4 \end{cases}$$

Here we use the alternative def^n of the derivative:

$$f'(4) = \lim_{x\to 4} \frac{f(x)-f(4)}{x-4}$$

$$= \lim_{x\to 4} \frac{f(x)-0}{x-4}$$

$$f'(4) = \lim_{x \to 4} \frac{f(x)}{x-4}$$

Because $f(x) = |x-4|$ changes its def'n at $x=4$, we consider the one-sided limits:

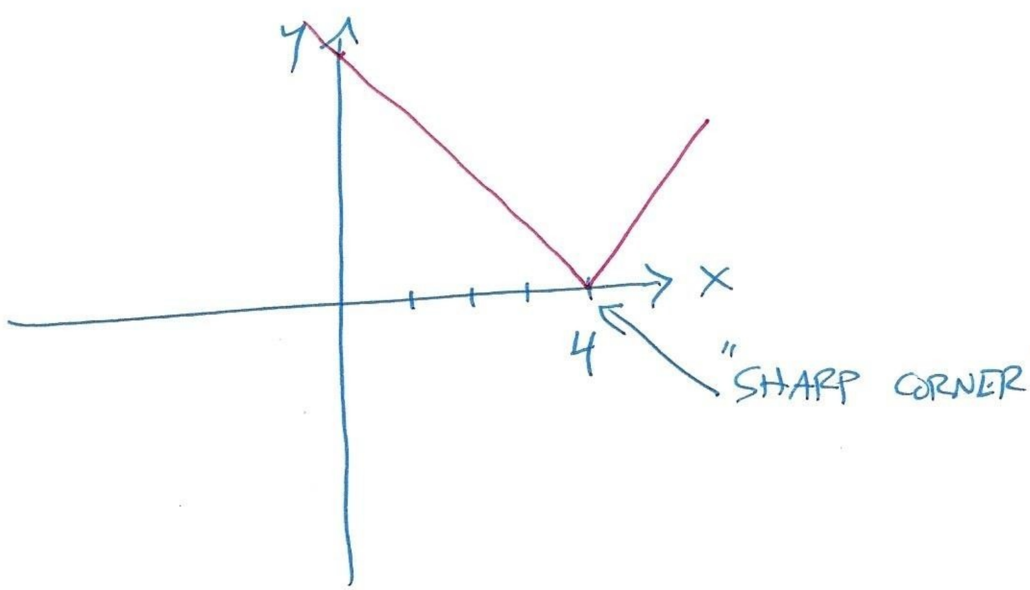
$$\lim_{x \to 4^-} \frac{f(x)}{x-4} = \lim_{x \to 4^-} \frac{-(x-4)}{x-4}$$

$$= \lim_{x \to 4^-} (-1) = -1$$

$$\lim_{x \to 4^+} \frac{f(x)}{x-4} = \lim_{x \to 4} \frac{x-4}{x-4}$$

$$= \lim_{x \to 4} 1 = 1$$

Thus the limit representing the derivative at $x=4$ does not exist, and so $f(x)$ is <u>non-differentiable</u> at $x=4$.



(3) If $f(x)$ is not continuous at $x=p$ then it is not differentiable at $x=p$.

In logic, we often study statements of the form "if A, then B".

The converse of this statement is "if (not A), then (not B)". The truth of the original statement does not guarantee the truth of its converse.

The contrapositive of the statement is "if (not B), then (not A)". The statement and its contrapositive must either both be true or both be false.

<u>Theorem</u>: If a function $f(x)$ is differentiable at $x=p$ then it is continuous at $x=p$.

<u>Proof</u>: For some function $f(x)$, we assume that it is differentiable at $x=p$, which means that

$$f'(p) = \lim_{x \to p} \frac{f(x)-f(p)}{x-p}$$

must exist.

We want to show that $f(x)$ is continuous at $x=p$. We can immediately conclude that $f(p)$ is defined since it appears in the limit for $f'(p)$. Thus we just need to show that

$$\lim_{x \to p} f(x) = f(p).$$

We can rewrite this as

$$\left[\lim_{x \to p} f(x)\right] - f(p) = 0$$

$$\left[\lim_{x \to p} f(x)\right] - \left[\lim_{x \to p} f(p)\right] = 0$$

$$\lim_{x \to p} \left[f(x) - f(p)\right] = 0$$

Observe that

$$f(x) - f(p) = \frac{f(x) - f(p)}{x-p} \cdot (x-p)$$

$$\lim_{x \to p} [f(x) - f(p)] = \lim_{x \to p} \left[\frac{f(x) - f(p)}{x-p} \cdot (x-p) \right]$$

$$= \left[\lim_{x \to p} \frac{f(x) - f(p)}{x-p} \right] \cdot \left[\lim_{x \to p} (x-p) \right]$$

$$= f'(p) \cdot (p-p)$$

$$= 0$$

Hence $f(x)$ is also continuous at $x = p$.