Section 2.1: Rates of Change

Given a function $y = f(x)$, this describes a process in which the <u>dependent</u> variable y changes as the <u>independent</u> variable x changes. We are often interested in the <u>rate of change</u> of y with respect to x.

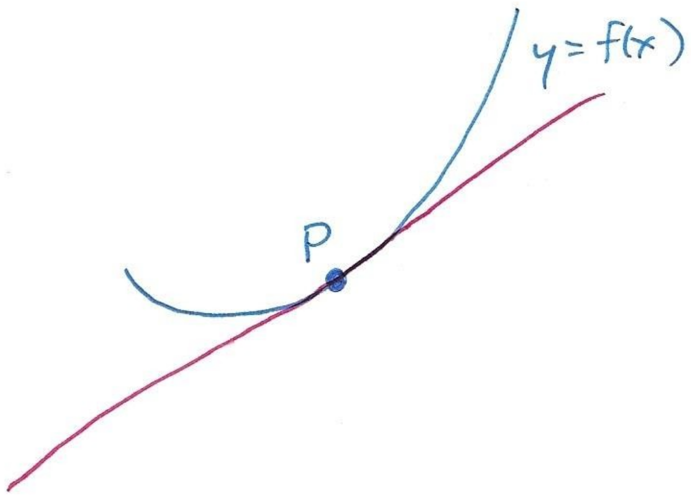
The slope of a line $y = mx + b$ is an example of a rate of change because

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are points on the line.

However, for any other function, the rate of change will not be the same at all points.

<u>Question</u>: Given a function $f(x)$ and a point P, how do we determine the rate of change of $f(x)$ at P?

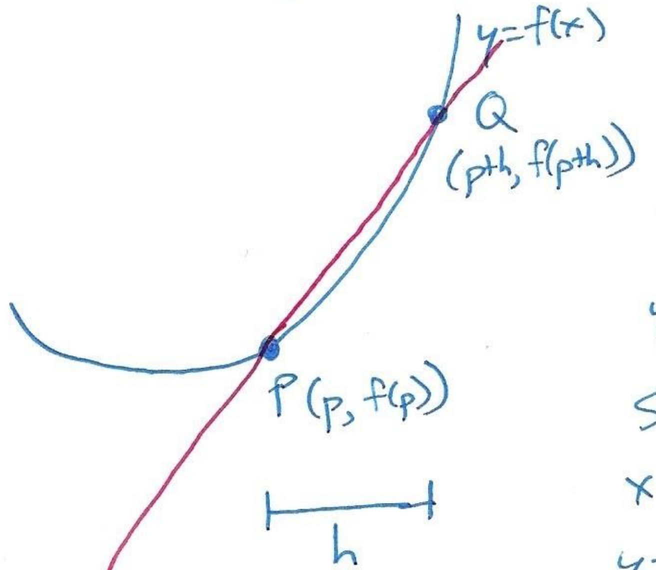


We draw a line which intersects the function $y=f(x)$ at P without passing through the graph at that point. This is a <u>tangent line</u>.

The rate of change of $f(x)$ at P and the slope of its tangent line at P must be equal.

Now we need two points on the tangent line in order to calculate its slope. Unfortunately, we only know that it passes through P.

Instead, we will pick a second point Q on the graph of $y=f(x)$ and draw a <u>secant line</u> which passes through both P and Q.



Suppose P is at $x=p$. Then its y-coordinate is $f(p)$.

Suppose Q is at $x=p+h$. Then its y-coordinate is $f(p+h)$.

The slope of the secant line is

$$M = \frac{f(p+h) - f(p)}{(p+h) - p} = \frac{f(p+h) - f(p)}{h}.$$

In the limit as $Q \to P$, the slope M of the secant line will approach the slope m of the tangent line. Thus

$$m = \lim_{Q \to P} \frac{f(p+h) - f(p)}{h}$$

$$= \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$

Direct substitution will always produce a $\frac{0}{0}$ form, so we will need our limit techniques from Unit 1 to calculate m.

Theorem: Given a function $f(x)$ and a real number p for which $f(x)$ is defined, the rate of change of $f(x)$ at $x=p$, and the slope of the tangent line to $y=f(x)$ at $x=p$, are given by

$$\lim_{h\to 0} \frac{f(p+h)-f(p)}{h}$$

eg Find the rate of change of $f(x) = 2x^2 - 1$ at the point $(2,7)$.

$$\lim_{h\to 0} \frac{f(p+h)-f(p)}{h} = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0} \frac{[2(2+h)^2-1]-7}{h}$$

$$= \lim_{h\to 0} \frac{2(4+4h+h^2)-8}{h}$$

$$= \lim_{h\to 0} \frac{8h+2h^2}{h}$$

$$= \lim_{h\to 0} \frac{2h(4+h)}{h}$$

$$= \lim_{h\to 0} 2(4+h)$$

$$\boxed{=8}$$

The slope of the tangent line to the curve $y = 2x^2 - 1$ at $(2,7)$ must be $m = 8$. In slope-intercept form, the equation of the tangent line would be

$$y = mx + b$$
$$y = 8x + b$$
$$7 = 8 \cdot 2 + b \rightarrow b = -9$$
$$\boxed{y = 8x - 9}$$

eg Find the equation of the tangent line to $f(x) = \sqrt[3]{x}$ at the point $(0,0)$.

The slope of the tangent line is

$$m = \lim_{h \to 0} \frac{f(p+h) - f(p)}{h}$$
$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt[3]{h} - 0}{h}$$
$$= \lim_{h \to 0} \frac{h^{1/3}}{h}$$
$$= \lim_{h \to 0} \frac{1}{h^{2/3}}$$

This limit does not exist because direct substitution yields a $\frac{k}{0}$ form.

Because a $\frac{k}{0}$ form implies that the one-sided limits are infinite, the tangent line must be a vertical line. Thus the equation of the tangent line is $\boxed{x = 0.}$

Kinematics is the study of an object in motion. We will consider motion only in a straight line. We will typically given a function $s(t)$ which represents the position of the object at time t. Position is always measured relative to a reference point where $s = 0$.

The velocity function $v(t)$ measures how quickly the object's position is changing at time t. Thus velocity is the rate of change of position.

eg A rocket is launched vertically upwards with a velocity of 80 ft/sec. Its height after t seconds is given by

$$s(t) = 80t - 16t^2.$$

What is the velocity of the rocket after exactly 2 seconds?

The <u>average</u> velocity of an object on an interval of time $t_0 \le t \le t_0 + h$ is

$$\bar{v} = \frac{s(t_0+h) - s(t_0)}{h}.$$

We will set $t_0 = 2$ and try different values of h.

For $h = 0.5$: $\quad \bar{v} = \dfrac{s(2.5) - s(2)}{0.5} = \dfrac{100 - 96}{0.5} = 8$

For $h = 0.1$: $\quad \bar{v} = \dfrac{s(2.1) - s(2)}{0.1} = \dfrac{97.44 - 96}{0.1} = 14.4$

For $h = 0.01$: $\quad \bar{v} = \dfrac{s(2.01) - s(2)}{0.01} = \dfrac{96.158 - 96}{0.01} = 15.84$

For $h = 0.001$: $\quad \bar{v} = \dfrac{s(2.001) - s(2)}{0.001} = \dfrac{96.016 - 96}{0.001} = 15.98$

We can conclude that

$$v(2) = \lim_{h \to 0} \bar{v} = \lim_{h \to 0} \frac{s(2+h) - s(2)}{h}$$

$$= \lim_{h \to 0} \frac{[80(2+h) - 16(2+h)^2] - 96}{h}$$

$$= \lim_{h \to 0} \frac{160 + 80h - 64 - 64h - 16h^2 - 96}{h}$$

$$= \lim_{h \to 0} \frac{16h - 16h^2}{h} = \lim_{h \to 0} (16 - 16h) = 16$$

Thus the instantaneous velocity at $t = 2$ is 16 ft/sec.