Section 1.7: Continuity on an Interval

Def'n: A function is continuous on an open interval (a,b) or $a < x < b$ if it is continuous at each point on that interval.

eg Given

$$f(x) = \begin{cases} \dfrac{x^2+6x+8}{x^2-4} & , \text{ for } x < -1 \\ x & , \text{ for } -1 \leq x < 3 \\ x^2-4 & , \text{ for } x \geq 3 \end{cases}$$

determine whether $f(x)$ is continuous for all real numbers $\mathbb{R}$ or find and classify any discontinuities.

① We will check to see if there are any points at which the different def'ns of $f(x)$ become undefined.

Both x and x^2-4 are polynomials, and hence defined everywhere. However, $\dfrac{x^2+6x+8}{x^2-4}$ is undefined if $x^2-4=0$

$$(x-2)(x+2)=0$$

$$x=2,\ x=-2$$

However, we do not use this def'n for $x=2$, so it can be omitted. Hence this step yields only $x=-2$ as a discontinuity.

To classify the discontinuity at $x=-2$, we evaluate

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x^2+6x+8}{x^2-4} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to -2} \frac{(x+2)(x+4)}{(x-2)(x+2)}$$

$$= \lim_{x\to -2} \frac{x+4}{x-2} = \frac{2}{-4} = -\frac{1}{2}$$

Hence $x=-2$ is a <u>removable</u> discontinuity.

(2) We will check all the points where the def'n of $f(x)$ changes. Here, this takes place at $x=-1$ and $x=3$.

At $x=-1$: $f(-1)=-1$

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \frac{x^2+6x+8}{x^2-4} = \frac{3}{-3}$$

$$= -1$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} x = -1$$

Hence $\lim_{x\to -1} f(x) = -1 = f(-1)$.

Thus $f(x)$ is continuous at $x=-1$.

At $x=3$: $f(3) = 3^2 - 4 = 5$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x = 3$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2 - 4) = 5$$

Hence $\lim_{x \to 3} f(x)$ does not exist.

Thus $x=3$ is a <u>non-removable</u> discontinuity.

<u>Def'n</u>: For any function $f(x)$ which is continuous on the open interval $a < x < b$ then

1. $f(x)$ is <u>continuous on the half-open interval</u> $(a, b]$ or $a < x \leq b$ if it is also left-continuous at $x = b$,
2. $f(x)$ is <u>continuous on the half-open interval</u> $[a, b)$ or $a \leq x < b$ if it is also right-continuous at $x = a$,
3. $f(x)$ is <u>continuous on the closed interval</u> $[a, b]$ or $a \leq x \leq b$ if it is also left-continuous at $x = b$ and right-continuous at $x = a$.

eg Consider $f(x) = \frac{1}{x}$ and determine whether it is continuous on the intervals $(0,5)$, $(0,5]$, $[0,5)$ and $[0,5]$.

The only potential discontinuity is at $x=0$. Since $x=0$ does not lie on $(0,5)$ or $(0,5]$ we know that $f(x)$ is continuous on these intervals. However, $f(x)$ is discontinuous on both $[0,5)$ and $[0,5]$.

<u>Theorem</u>: If both $f(x)$ and $g(x)$ are continuous at $x=p$ then the following are also continuous functions at $x=p$:

1. $f(x)+g(x)$
2. $f(x)-g(x)$
3. $kf(x)$ for any constant k
4. $f(x)g(x)$
5. $\dfrac{f(x)}{g(x)}$ if $g(p) \neq 0$
6. if $g(x)$ is continuous at $x=p$ and $f(x)$ is continuous at $x=g(p)$ then $(f \circ g)(x) = f(g(x))$ is continuous at $x=p$.

Theorem: If $f(x)$ is continuous at $x=q$ and $g(x)$ is a function for which $\lim_{x \to p} g(x) = q$ then

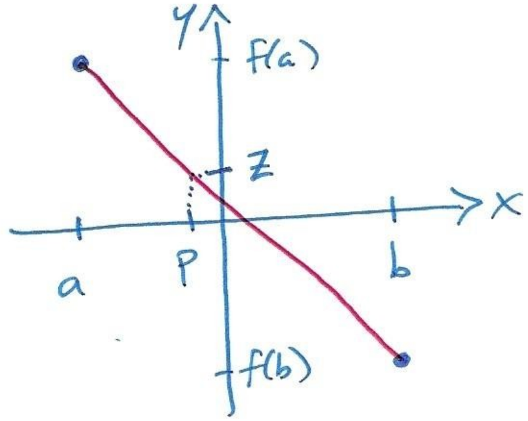
$$\lim_{x \to p} (f \circ g)(x) = \lim_{x \to p} f(g(x))$$
$$= f\left(\lim_{x \to p} g(x)\right)$$
$$= f(q).$$

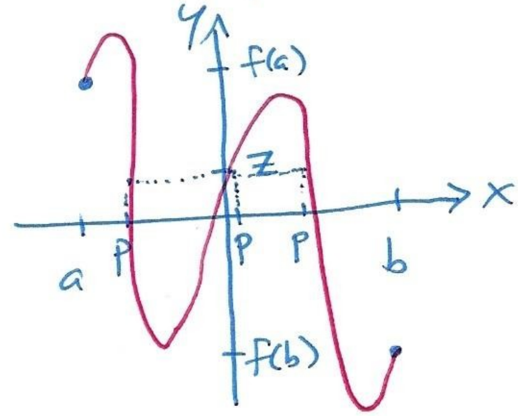
eg $\lim_{x \to 0} \cos(x^2)$

Since $\cos(x)$ is continuous everywhere, we can write

$$\lim_{x \to 0} \cos(x^2) = \cos\left(\lim_{x \to 0} x^2\right)$$
$$= \cos(0)$$
$$\boxed{= 1}$$

The Intermediate Value Theorem: Suppose that $f(x)$ is continuous on the closed interval $a \le x \le b$, and that $f(a) \neq f(b)$. Let z be any number between $f(a)$ and $f(b)$. Then there exists a number p in the open interval $a < x < b$ such that $f(p) = z$.





eg Show that $f(x) = x^3 + 2x^2 - 4x - 1$ has a root between $x=-1$ and $x=1$.

We want to show that there exists at least one point $x=p$ on the interval $-1<p<1$ for which $f(p)=0$.

Since $f(x)$ is a polynomial function, it is everywhere continuous, and thus it is certainly continuous on $-1 \le x \le 1$. Also,

$$f(1) = -2 \quad \text{and} \quad f(-1) = 4$$

Since $-2 < 0 < 4$, we conclude by the IVT that there exists a point $x=p$ on this interval where $f(p)=0$