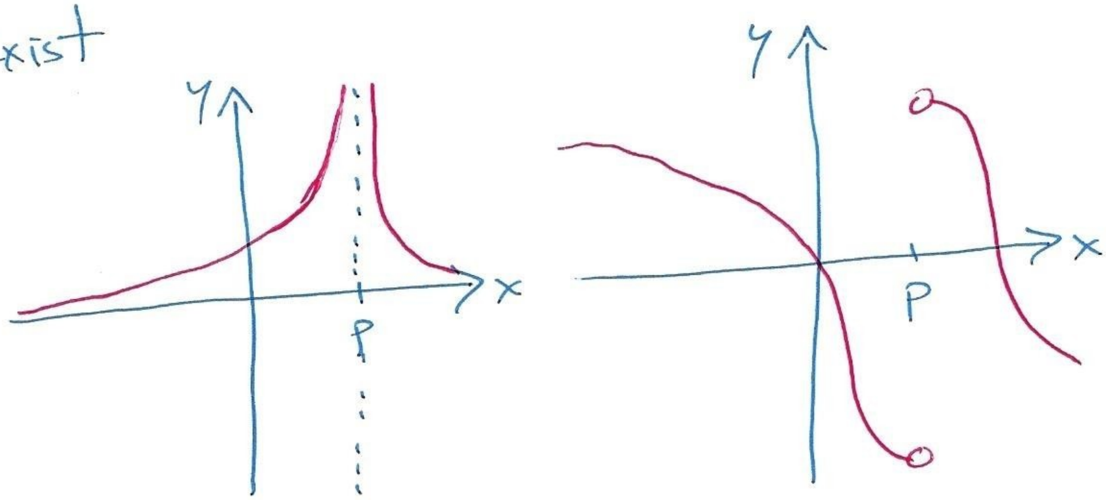
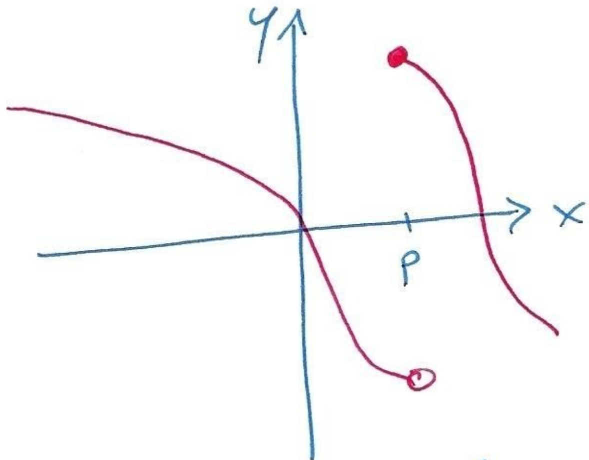
Section 1.6: Continuity

Given a function $f(x)$ and a point $x=p$, there are several possible relationships between $f(p)$ and $\lim_{x\to p} f(x)$.

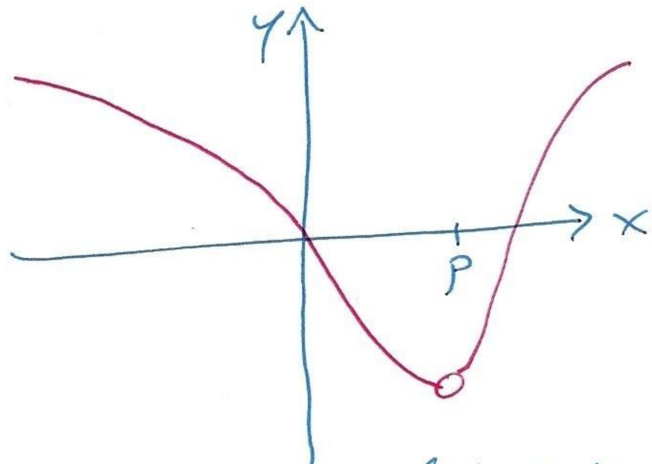
① $f(p)$ is undefined and $\lim_{x\to p} f(x)$ does not exist



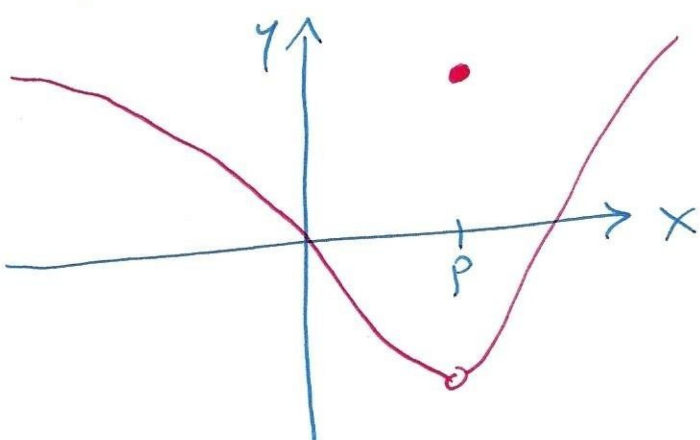
② $f(p)$ is defined but $\lim_{x \to p} f(x)$ does not exist



③ $f(p)$ is undefined but $\lim_{x \to p} f(x)$ exists

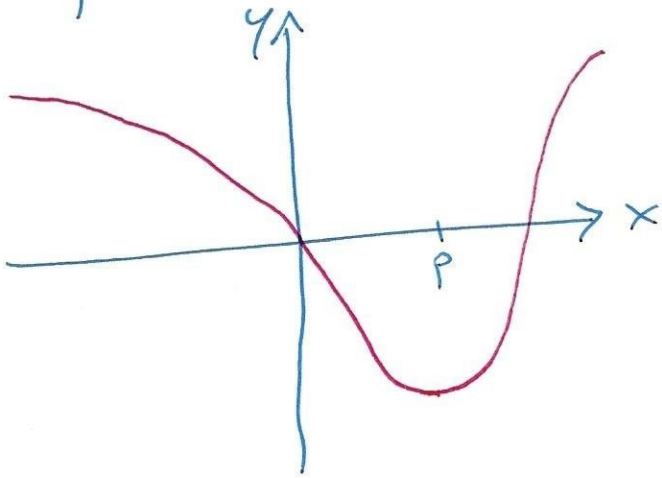


④ $f(p)$ is defined and $\lim_{x \to p} f(x)$ exists but $f(p) \neq \lim_{x \to p} f(x)$



In each of cases 1 to 4, we say that $f(x)$ is <u>discontinuous</u> at $x=p$.

5 $f(p)$ is defined, $\lim_{x\to p} f(x)$ exists <u>and</u>
$f(p) = \lim_{x\to p} f(x)$



<u>Def'n</u>: A function $f(x)$ is <u>continuous at a point</u> $x=p$ if each of the following hold:

1. $f(p)$ is defined
2. $\lim_{x\to p} f(x)$ exists
3. $f(p) = \lim_{x\to p} f(x)$

eg Determine whether $g(x)$ is continuous at $x=3$ given

$$g(x) = \begin{cases} x-3, & \text{for } x<3 \\ 4, & \text{for } x=3 \\ \sqrt{x-3}, & \text{for } x>3. \end{cases}$$

$g(3) = 4$

$\lim_{x \to 3^-} g(x) = \lim_{x \to 3^-} (x-3) = 0$

$\lim_{x \to 3^+} g(x) = \lim_{x \to 3^+} \sqrt{x-3} = 0$

Thus $\lim_{x \to 3} g(x) = 0$.

However, $g(3) \neq \lim_{x \to 3} g(x)$ so $g(x)$ is <u>discontinuous</u> at $x=3$.

eg Given $f(x) = \begin{cases} kx^2 - 1, & \text{for } x \leq 2 \\ x + k, & \text{for } x > 2 \end{cases}$

find all values of k for which $f(x)$ is continuous at $x = 2$.

$f(2) = 4k - 1$ which is defined for all k

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (kx^2 - 1) = 4k - 1$$

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x + k) = 2 + k$$

To ensure that $\lim_{x \to 2} f(x)$ exists, we require

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$4k - 1 = 2 + k$$

$$3k = 3$$

$$k = 1$$

For $k = 1$, $\lim_{x \to 2} f(x) = 3$, while $f(2) = 3$.

Hence $f(2) = \lim_{x \to 2} f(x)$.

Thus $f(x)$ is continuous at $x = 2$ only for $\boxed{k = 1}$.

Def'n: Given a function $f(x)$, we say that a discontinuity $x=p$ is removable if $\lim_{x \to p} f(x)$ exists. Otherwise, it is non-removable.

Def'n: A non-removable discontinuity at $x=p$ in the graph of a function $f(x)$ is called a jump discontinuity if $\lim_{x \to p^-} f(x)$ and $\lim_{x \to p^+} f(x)$ both exist but $\lim_{x \to p^-} f(x) \neq \lim_{x \to p^+} f(x)$.

Def'n: A non-removable discontinuity at $x=p$ in the graph of a function $f(x)$ is called an infinite discontinuity if both of the one-sided limits as $x \to p$ are infinite limits. Thus an infinite discontinuity occurs at a vertical asymptote $x=p$.

eg Given
$$f(x) = \begin{cases} \dfrac{x^3+x^2-2x}{1-x}, & \text{for } x \neq 1 \\ 2, & \text{for } x = 1 \end{cases}$$

investigate the continuity of $f(x)$ at $x=1$.

$f(1) = 2$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^3 + x^2 - 2x}{1 - x} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 1} \frac{x(x^2 + x - 2)}{-(x-1)}$$

$$= \lim_{x \to 1} \frac{x(x-1)(x+2)}{-(x-1)}$$

$$= \lim_{x \to 1} \frac{x(x+2)}{-1} = -3$$

Since $f(1) \neq \lim_{x \to 1} f(x)$, $f(x)$ is <u>discontinuous</u> at $x = 1$. Since $\lim_{x \to 1} f(x)$ exists, this is a <u>removable</u> discontinuity.

Def'n: A function $f(x)$ is <u>left-continuous</u> at a point $x=p$ if each of the following holds:

① $f(p)$ is defined

② $\lim_{x \to p^-} f(x)$ exists

③ $f(p) = \lim_{x \to p^-} f(x)$

Def'n: A function $f(x)$ is <u>right-continuous</u> at a point $x=p$ if each of the following holds:

① $f(p)$ is defined

② $\lim_{x \to p^+} f(x)$ exists

③ $f(p) = \lim_{x \to p^+} f(x)$

Theorem: A function is continuous at a point $x=p$ if and only if it is both left-continuous and right-continuous at $x=p$.