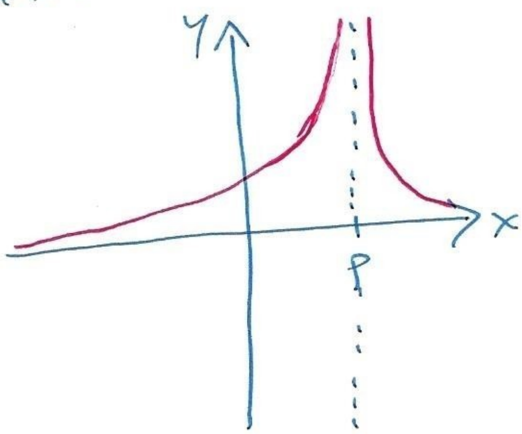
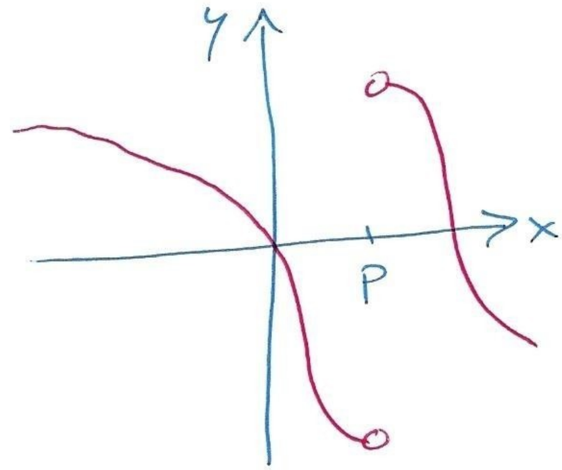
Section 1.6: Continuity

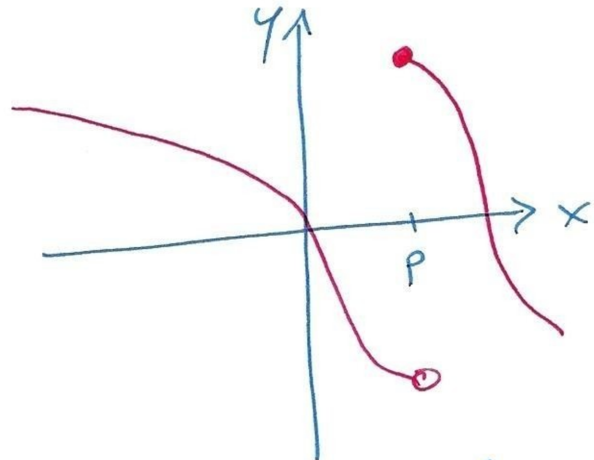
Given a function $f(x)$ and a point $x = p$, there are several possible relationships between $f(p)$ and $\lim_{x \to p} f(x)$.

① $f(p)$ is undefined and $\lim_{x \to p} f(x)$ does not exist

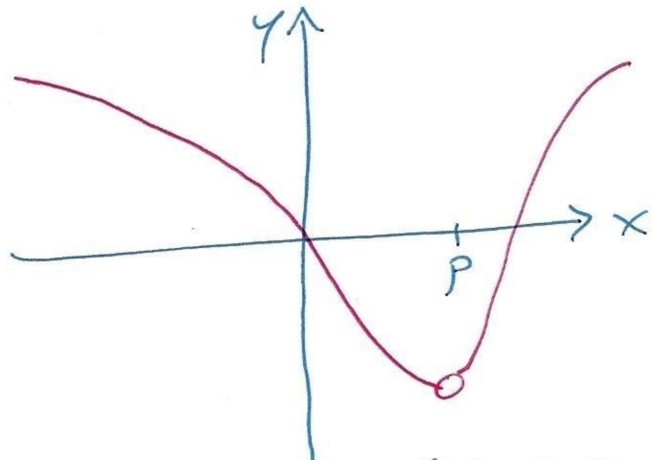




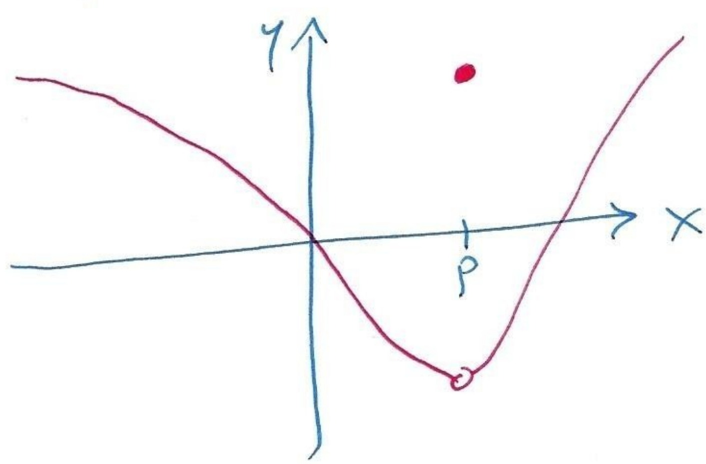
② $f(p)$ is defined but $\lim_{x \to p} f(x)$ does not exist



③ $f(p)$ is undefined but $\lim_{x \to p} f(x)$ exists

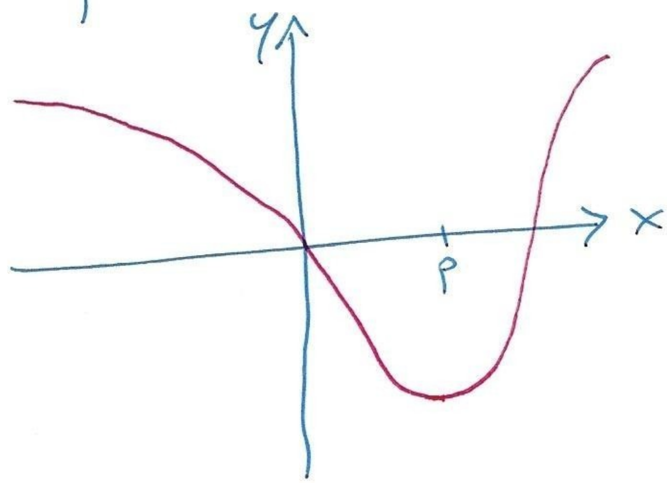


④ $f(p)$ is defined and $\lim_{x \to p} f(x)$ exists but $f(p) \neq \lim_{x \to p} f(x)$



In each of cases ① to ④, we say that $f(x)$ is <u>discontinuous</u> at $x=p$.

⑤ $f(p)$ is defined, $\lim_{x\to p} f(x)$ exists <u>and</u> $f(p) = \lim_{x\to p} f(x)$



<u>Def'n</u>: A function $f(x)$ is <u>continuous at a point</u> $x=p$ if each of the following hold:

① $f(p)$ is defined

② $\lim_{x\to p} f(x)$ exists

③ $f(p) = \lim_{x\to p} f(x)$

eg Determine whether $f(x)$ is continuous at $x=3$ given

$$f(x) = \begin{cases} x-3, & \text{for } x<3 \\ \sqrt{x-3}, & \text{for } x>3 \end{cases}$$

$f(3)$ is undefined, so $f(x)$ is <u>discontinuous</u> at $x=3$.

eg Determine whether $g(x)$ is continuous at $x=3$ given

$$g(x) = \begin{cases} x-3, & \text{for } x<3 \\ 4, & \text{for } x=3 \\ \sqrt{x-3}, & \text{for } x>3. \end{cases}$$

$g(3)=4$

$\lim\limits_{x\to 3^-} g(x) = \lim\limits_{x\to 3^-} (x-3) = 0$

$\lim\limits_{x\to 3^+} g(x) = \lim\limits_{x\to 3^+} \sqrt{x-3} = 0$

Thus $\lim\limits_{x\to 3} g(x) = 0$.

However, $g(3) \neq \lim\limits_{x\to 3} g(x)$ so $g(x)$ is <u>discontinuous</u> at $x=3$.

eg Determine whether $h(x)$ is con......

$x=3$ given

$$h(x) = \begin{cases} x-3, & \text{for } x \leq 3 \\ \sqrt{x-3}, & \text{for } x>3. \end{cases}$$

$h(3)=0$

$\lim\limits_{x\to 3^-} h(x) = \lim\limits_{x\to 3^-} (x-3) = 0$

$\lim\limits_{x\to 3^+} h(x) = \lim\limits_{x\to 3^+} \sqrt{x-3} = 0$

So $\lim\limits_{x\to 3} h(x) = 0 = h(3)$.

Thus $h(x)$ <u>is continuous</u> at $x=3$.

eg Given $f(x) = \begin{cases} kx^2 - 1, & \text{for } x \leq 2 \\ x + k, & \text{for } x > 2 \end{cases}$

find all values of k for which $f(x)$ is continuous at $x = 2$.

$f(2) = 4k - 1$ which is defined for all k

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (kx^2 - 1) = 4k - 1$

$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x + k) = 2 + k$

To ensure that $\lim_{x \to 2} f(x)$ exists, we require

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$4k - 1 = 2 + k$$

$$3k = 3$$

$$k = 1$$

For $k = 1$, $\lim_{x \to 2} f(x) = 3$, while $f(2) = 3$.

Hence $f(2) = \lim_{x \to 2} f(x)$.

Thus $f(x)$ is continuous at $x = 2$ only for $\boxed{k = 1}$.

Def'n: Given a function $f(x)$, we say that a discontinuity $x=p$ is <u>removable</u> if $\lim_{x \to p} f(x)$ exists. Otherwise, it is <u>non-removable</u>.

Def'n: A non-removable discontinuity at $x=p$ in the graph of a function $f(x)$ is called a <u>jump discontinuity</u> if $\lim_{x \to p^-} f(x)$ and $\lim_{x \to p^+} f(x)$ both exist but $\lim_{x \to p^-} f(x) \neq \lim_{x \to p^+} f(x)$.

Def'n: A non-removable discontinuity at $x=p$ in the graph of a function $f(x)$ is called an <u>infinite discontinuity</u> if both of the one-sided limits as $x \to p$ are infinite limits. Thus an infinite discontinuity occurs at a vertical asymptote $x=p$.

eg Given
$$f(x) = \begin{cases} \dfrac{x^3+x^2-2x}{1-x}, & \text{for } x \neq 1 \\ 2, & \text{for } x = 1 \end{cases}$$

investigate the continuity of $f(x)$ at $x=1$.

$f(1) = 2$

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^3+x^2-2x}{1-x} \qquad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 1} \frac{x(x^2+x-2)}{-(x-1)}$$

$$= \lim_{x\to 1} \frac{x(x-1)(x+2)}{-(x-1)}$$

$$= \lim_{x\to 1} \frac{x(x+2)}{-1} = -3$$

Since $f(1) \neq \lim_{x\to 1} f(x)$, $f(x)$ is <u>discontinuous</u> at $x=1$. Since $\lim_{x\to 1} f(x)$ exists, this is a <u>removable</u> discontinuity.

The <u>greatest integer</u> or <u>floor</u> function, denoted by $[\![x]\!]$ or $\lfloor x \rfloor$, takes any real number x and outputs the greatest integer that is equal to or less than x.

eg $\lfloor 3 \rfloor = 3$ $\qquad$ $\lfloor 0.4 \rfloor = 0$

$\lfloor 7.8 \rfloor = 7$ $\qquad$ $\lfloor -2.1 \rfloor = -3$

Def'n: A function $f(x)$ is <u>left-continuous</u> at a point $x=p$ if each of the following holds:

1. $f(p)$ is defined
2. $\lim_{x \to p^-} f(x)$ exists
3. $f(p) = \lim_{x \to p^-} f(x)$

Def'n: A function $f(x)$ is <u>right-continuous</u> at a point $x=p$ if each of the following holds:

1. $f(p)$ is defined
2. $\lim_{x \to p^+} f(x)$ exists
3. $f(p) = \lim_{x \to p^+} f(x)$

Theorem: A function is continuous at a point $x=p$ if and only if it is both left-continuous and right-continuous at $x=p$.

Def'n : A function is continuous on an open interval (a,b) or $a < x < b$ if it is continuous at each point on that interval.

eg Given

$$f(x) = \begin{cases} \dfrac{x^2+6x+8}{x^2-4}, & \text{for } x < -1 \\ x, & \text{for } -1 \leq x < 3 \\ x^2-4, & \text{for } x \geq 3 \end{cases}$$

determine whether $f(x)$ is continuous for all real numbers $\mathbb{R}$ or find and classify any discontinuities.

① We will check to see if there are any points at which the different def'ns of $f(x)$ become undefined.

Both x and x^2-4 are polynomials, and hence defined everywhere. However, $\dfrac{x^2+6x+8}{x^2-4}$ is undefined if $x^2-4=0$

$$(x-2)(x+2)=0$$

$$x=2,\ x=-2$$

However, we do not use this def'n for $x=2$, so it can be omitted. Hence this step yields only $x=-2$ as a discontinuity.

To classify the discontinuity at $x=-2$, we evaluate

$$\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2+6x+8}{x^2-4} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to -2} \frac{(x+2)(x+4)}{(x-2)(x+2)}$$

$$= \lim_{x \to -2} \frac{x+4}{x-2} = \frac{2}{-4} = -\frac{1}{2}$$

Hence $x=-2$ is a <u>removable</u> discontinuity.

(2) We will check all the points where the def'n of $f(x)$ changes. Here, this takes place at $x=-1$ and $x=3$.

At $x=-1$: $f(-1) = -1$

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} \frac{x^2+6x+8}{x^2-4} = \frac{3}{-3}$$
$$= -1$$

$$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} x = -1$$

Hence $\lim_{x \to -1} f(x) = -1 = f(-1)$.

Thus $f(x)$ is continuous at $x=-1$.

At $x=3$: $f(3) = 3^2 - 4 = 5$

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} x = 3$$

$$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x^2 - 4) = 5$$

Hence $\lim_{x \to 3} f(x)$ does not exist.

Thus $x = 3$ is a non-removable discontinuity.

Def'n: For any function $f(x)$ which is continuous on the open interval $a < x < b$ then

1. $f(x)$ is continuous on the half-open interval $(a, b]$ or $a < x \leq b$ if it is also left-continuous at $x = b$,
2. $f(x)$ is continuous on the half-open interval $[a, b)$ or $a \leq x < b$ if it is also right-continuous at $x = a$,
3. $f(x)$ is continuous on the closed interval $[a, b]$ or $a \leq x \leq b$ if it is also left-continuous at $x = b$ and right-continuous at $x = a$.

eg Consider $f(x) = \frac{1}{x}$ and determine whether it is continuous on the intervals $(0,5)$, $(0,5]$, $[0,5)$ and $[0,5]$.

The only potential discontinuity is at $x=0$. Since $x=0$ does not lie on $(0,5)$ or $(0,5]$ we know that $f(x)$ is continuous on these intervals. However, $f(x)$ is discontinuous on both $[0,5)$ and $[0,5]$.

Theorem: If both $f(x)$ and $g(x)$ are continuous at $x=p$ then the following are also continuous functions at $x=p$:

1. $f(x)+g(x)$
2. $f(x)-g(x)$
3. $kf(x)$ for any constant k
4. $f(x)g(x)$
5. $\dfrac{f(x)}{g(x)}$ if $g(p) \neq 0$
6. if $g(x)$ is continuous at $x=p$ and $f(x)$ is continuous at $x=g(p)$ then $(f \circ g)(x) = f(g(x))$ is continuous at $x=p$.

Theorem: If $f(x)$ is continuous at $x = q$ and $g(x)$ is a function for which $\lim_{x \to p} g(x) = q$ then

$$\lim_{x \to p} (f \circ g)(x) = \lim_{x \to p} f(g(x))$$

$$= f\left(\lim_{x \to p} g(x)\right)$$

$$= f(q).$$

eg $\lim_{x \to 0} \cos(x^2)$

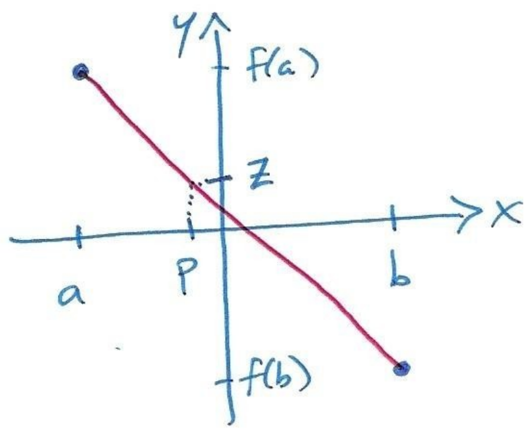
Since $\cos(x)$ is continuous everywhere, we can write

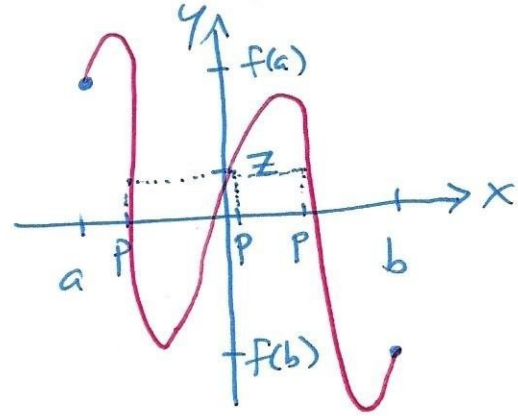
$$\lim_{x \to 0} \cos(x^2) = \cos\left(\lim_{x \to 0} x^2\right)$$

$$= \cos(0)$$

$$\boxed{= 1}$$

The Intermediate Value Theorem: Suppose that $f(x)$ is continuous on the closed interval $a \le x \le b$, and that $f(a) \neq f(b)$. Let z be any number between $f(a)$ and $f(b)$. Then there exists a number p in the open interval $a < x < b$ such that $f(p) = z$.





eg Show that $f(x) = x^3 + 2x^2 - 4x - 1$ has a root between $x = -1$ and $x = 1$.

We want to show that there exists at least one point $x = p$ on the interval $-1 < p < 1$ for which $f(p) = 0$.

Since $f(x)$ is a polynomial function, it is everywhere continuous, and thus it is certainly continuous on $-1 \le x \le 1$. Also,

$$f(1) = -2 \quad \text{and} \quad f(-1) = 4$$

Since $-2 < 0 < 4$, we conclude by the IVT that there exists a point $x = p$ on this interval where $f(p) = 0$