Section 1.5: Limits at Infinity

<u>Def'n</u>: Let L be a real number. If $f(x)$ can be made arbitrarily close to L by letting x increase without bound then we write $\lim_{x \to \infty} f(x) = L$.

If $f(x)$ can be made arbitrarily close to L by letting x decrease without bound then we write $\lim_{x \to -\infty} f(x) = L$.

Collectively, these are the <u>limits at infinity</u>.

All 5 of the Basic Limit Properties apply to the limits at infinity as well.

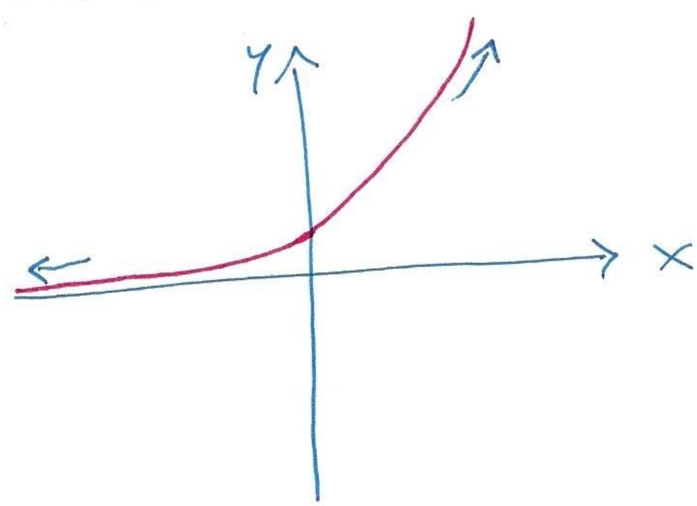
eg $$\lim_{x\to\infty}[f(x)+g(x)] = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x)$$

$$\lim_{x\to-\infty}[f(x)+g(x)] = \lim_{x\to-\infty} f(x) + \lim_{x\to-\infty} g(x)$$

Also, $\lim_{x\to\infty} k = k$ and $\lim_{x\to-\infty} k = k$ for any constant k.

For many common functions, such as polynomial and trigonometric functions, the limits at infinity do <u>not</u> exist. But there are exceptions.

eg Consider $f(x) = 2^x$.



$$\lim_{x\to\infty} 2^x = \infty$$

$$\lim_{x\to-\infty} 2^x = 0$$

<u>Def'n</u>: The line $y = L$ is called a <u>horizontal asymptote</u> to the graph of $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L.$$

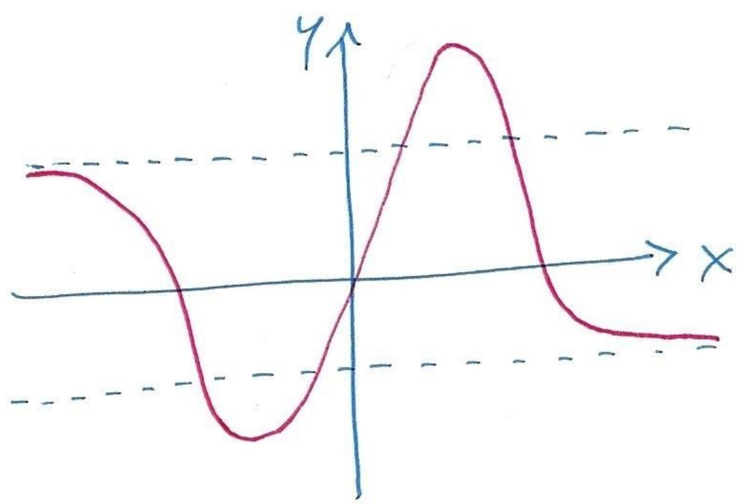
eg $f(x) = 2^x$

Because $\lim_{x \to -\infty} 2^x = 0$, $y = 0$ is a horizontal asymptote.

Note that a function may have any number of vertical asymptotes, but only 0, 1 or 2 horizontal asymptotes.

Also, while a function <u>cannot</u> cross through a vertical asymptote, but <u>can</u> cross through a horizontal asymptote any number of times.

eg



Theorem: If r is a positive real number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

and $\displaystyle\lim_{x\to-\infty} \frac{1}{x^r} = 0$ if x^r is defined for $x<0$.

We can use this result to compute limits at infinity of rational functions.

eg $\displaystyle\lim_{x\to\infty} \frac{2x^2+x-1}{x^2-6}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

We will divide the numerator and the denominator by the largest power of x in the denominator. Here, this is x^2:

$$\lim_{x\to\infty} \frac{2x^2+x-1}{x^2-6} \cdot \frac{1/x^2}{1/x^2}$$

$$= \lim_{x\to\infty} \frac{2+1/x-1/x^2}{1-6/x^2}$$

$$= \frac{2+0-0}{1-0} \boxed{=2}$$

Hence $y=2$ is a horizontal asymptote.

eg $\lim_{x \to \infty} \frac{4-6x^3}{2x^2-x-1} \cdot \frac{1/x^2}{1/x^2}$

$= \lim_{x \to \infty} \frac{4/x^2 - 6x}{2 - 1/x - 1/x^2}$

$= \frac{0-\infty}{2-0-0} \boxed{= -\infty}$

Theorem: If $f(x)$ is a rational function and L is a real number then

$$\lim_{x \to \infty} f(x) = L \text{ if and only if } \lim_{x \to -\infty} f(x) = L.$$

Hence a rational function has either 0 or 1 horizontal asymptote.

eg Find any horizontal asymptotes to the graph of $f(x) = \dfrac{5x^3+1}{(2x+1)(x^2+3)}$.

We must find

$$\lim_{x\to\infty} \frac{5x^3+1}{(2x+1)(x^2+3)}$$

$$= \lim_{x\to\infty} \frac{5x^3+1}{2x^3+x^2+6x+3} \cdot \frac{1/x^3}{1/x^3}$$

$$= \lim_{x\to\infty} \frac{5 + 1/x^3}{2 + 1/x + 6/x^2 + 3/x^3}$$

$$= \frac{5+0}{2+0+0+0} = \frac{5}{2}$$

Hence $\lim_{x\to-\infty} f(x) = \frac{5}{2}$ as well.

The only horizontal asymptote is $\boxed{y = \frac{5}{2}.}$

For limits at infinity of quasirational functions, first recall that

$$\sqrt{x^2} = \begin{cases} x, & \text{for } x \geq 0 \\ -x, & \text{for } x < 0 \end{cases} = |x|$$

Thus $x = \sqrt{x^2}$ if $x \geq 0$ but $x = -\sqrt{x^2}$ if $x < 0$.

eg Find any horizontal asymptotes of

$$f(x) = \frac{6x}{3x + \sqrt{x^2+2x-1}}$$

We identify the dominant power of x in the denominator, treating any power under the square root as if it has ½ of its value. Here, this is $x^1 = x$.

Now we have

$$\lim_{x\to\infty} \frac{6x}{3x+\sqrt{x^2+2x-1}} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x\to\infty} \frac{6}{3+\frac{1}{x}\sqrt{x^2+2x-1}}$$

$$= \lim_{x\to\infty} \frac{6}{3+\frac{1}{\sqrt{x^2}}\sqrt{x^2+2x-1}}$$

$$= \lim_{x\to\infty} \frac{6}{3+\sqrt{1+\frac{2}{x}-\frac{1}{x^2}}}$$

$$= \frac{6}{3+\sqrt{1+0-0}} = \frac{6}{4} = \frac{3}{2}$$

Thus $y = \frac{3}{2}$ is a horizontal asymptote.

Next,

$$\lim_{x\to-\infty} \frac{6x}{3x+\sqrt{x^2+2x-1}} \cdot \frac{1/x}{1/x}$$

$$= \lim_{x\to-\infty} \frac{6}{3+\frac{1}{x}\sqrt{x^2+2x-1}}$$

$$= \lim_{x \to -\infty} \frac{6}{3 + \frac{1}{-\sqrt{x^2}} \sqrt{x^2 + 2x - 1}}$$

$$= \lim_{x \to -\infty} \frac{6}{3 - \sqrt{1 + \frac{2}{x} - \frac{1}{x^2}}}$$

$$= \frac{6}{3 - \sqrt{1 + 0 - 0}} = \frac{6}{2} = 3$$

Thus $y = 3$ is also a horizontal asymptote.