# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

ASSIGNMENT 4 | **MATHEMATICS 1000** | FALL 2022

**SOLUTIONS**

[5] 1. By the limit definition,

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{\frac{4(x+h)}{(x+h)-3} - \frac{4x}{x-3}}{h} \\
&= \lim_{h\to 0} \left[ \frac{4(x+h)(x-3)}{(x-3)(x+h-3)} - \frac{4x(x+h-3)}{(x-3)(x+h-3)} \right] \cdot \frac{1}{h} \\
&= \lim_{h\to 0} \frac{4x^2 + 4hx - 12x - 12h - 4x^2 - 4hx + 12x}{h(x-3)(x+h-3)} \\
&= \lim_{h\to 0} \frac{-12h}{h(x-3)(x+h-3)} \\
&= \lim_{h\to 0} \frac{-12}{(x-3)(x+h-3)} \\
&= \frac{-12}{(x-3)^2}.
\end{aligned}$$

[7] 2. By the limit definition,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sqrt{(x+h)+4} - \sqrt{x+4}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h+4} - \sqrt{x+4}}{h} \cdot \frac{\sqrt{x+h+4} + \sqrt{x+4}}{\sqrt{x+h+4} + \sqrt{x+4}} \\
&= \lim_{h\to 0} \frac{(x+h+4) - (x+4)}{h\left[\sqrt{x+h+4} + \sqrt{x+4}\right]} \\
&= \lim_{h\to 0} \frac{h}{h\left[\sqrt{x+h+4} + \sqrt{x+4}\right]} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h+4} + \sqrt{x+4}} \\
&= \frac{1}{2\sqrt{x+4}}.
\end{aligned}$$

At $x = -3$, the slope of the tangent line is

$$f'(-3) = \frac{1}{2\sqrt{-3+4}} = \frac{1}{2}.$$

The $y$-coordinate when $x = -3$ is

$$f(-3) = \sqrt{-3+4} = 1.$$

Thus the equation of the tangent line has the form

$$\begin{aligned} y &= \frac{1}{2}x + b \\ 1 &= \frac{1}{2}(-3) + b \\ \frac{5}{2} &= b, \end{aligned}$$

and so the equation of the tangent line is

$$y = \frac{1}{2}x + \frac{5}{2}.$$

[4] 3. (a) Note that

$$f(1) = 3(1^2) - 4(1) + 1 = 0.$$

We use the alternative definition of the derivative

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{f(x) - 0}{x - 1} = \lim_{x \to 1} \frac{f(x)}{x - 1}$$

and consider the one-sided limits. From the left,

$$\begin{aligned} \lim_{x \to 1^-} \frac{f(x)}{x-1} &= \lim_{x \to 1^-} \frac{4x^2 - 8x + 4}{x^2 - 1} \cdot \frac{1}{x - 1} \\ &= \lim_{x \to 1^-} \frac{4(x^2 - 2x + 1)}{(x-1)(x^2-1)} \\ &= \lim_{x \to 1^-} \frac{4(x-1)^2}{(x-1)^2(x+1)} \\ &= \lim_{x \to 1^-} \frac{4}{x+1} \\ &= 2. \end{aligned}$$

From the right,

$$\begin{aligned} \lim_{x \to 1^+} \frac{f(x)}{x-1} &= \lim_{x \to 1^+} \frac{3x^2 - 4x + 1}{x - 1} \\ &= \lim_{x \to 1^+} \frac{(x-1)(3x-1)}{x-1} \\ &= \lim_{x \to 1^+} (3x - 1) \\ &= 2. \end{aligned}$$

Since the one-sided limits are equal, $f'(1) = 2$ and so $f(x)$ is differentiable at $x = 1$.

[4] (b) Note that

$$g(1) = 1 - 1 = 0.$$

We again use the alternative definition of the derivative

$$g'(1) = \lim_{x\to 1} \frac{g(x) - g(1)}{x - 1} = \lim_{x\to 1} \frac{g(x) - 0}{x - 1} = \lim_{x\to 1} \frac{g(x)}{x - 1}$$

and consider the one-sided limits. From the left,

$$\begin{aligned}
\lim_{x\to 1^-} \frac{g(x)}{x-1} &= \lim_{x\to 1^-} \frac{4x^2 - 8x + 4}{x^2 - 1} \cdot \frac{1}{x-1} \\
&= \lim_{x\to 1^-} \frac{4(x^2 - 2x + 1)}{(x-1)(x^2-1)} \\
&= \lim_{x\to 1^-} \frac{4(x-1)^2}{(x-1)^2(x+1)} \\
&= \lim_{x\to 1^-} \frac{4}{x+1} \\
&= 2.
\end{aligned}$$

From the right,

$$\begin{aligned}
\lim_{x\to 1^+} \frac{g(x)}{x-1} &= \lim_{x\to 1^+} \frac{1-x}{x-1} \\
&= \lim_{x\to 1^+} \frac{-(x-1)}{x-1} \\
&= \lim_{x\to 1^+} (-1) \\
&= -1.
\end{aligned}$$

This time, the one-sided limits are not equal, and so $g'(1)$ is not defined. Hence $g(x)$ is not differentiable at $x = 1$.