MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 4.1 | **Math 1000 Worksheet** | FALL 2022

SOLUTIONS

1. Let the length of the legs of the triangle be a and the length of the hypotenuse be h. We are given that $\dfrac{da}{dt} = 1$ and we want to find $\dfrac{dh}{dt}$ when $h = 5$ cm; to be consistent with the units, we must write this as $h = 50$ mm. Since the triangle is a right triangle, we can use the Pythagorean theorem, which for an isosceles right triangle is simply

$$2a^2 = h^2 \tag{$\dagger$}$$

so that

$$\begin{aligned} 4a\frac{da}{dt} &= 2h\frac{dh}{dt} \\ 2a\frac{da}{dt} &= h\frac{dh}{dt}. \end{aligned} \tag{$\star$}$$

To find $\dfrac{dh}{dt}$ we need to know a, which can be determined using Equation ($\dagger$):

$$2a^2 = 50^2 \quad \Longrightarrow \quad a = \sqrt{1250} = 25\sqrt{2}.$$

Hence, using Equation ($\star$),

$$2\left(25\sqrt{2}\right)(1) = 50\frac{dh}{dt} \quad \Longrightarrow \quad \frac{dh}{dt} = \sqrt{2}.$$

The hypotenuse is getting larger at a rate of $\sqrt{2}$ mm/sec.

2. Let y be the vertical height of the kite, x be the horizontal distance between the girl and the kite, ℓ be the length of the string, and θ be the angle between the string and the ground. We are given that $y = 100$ and $\dfrac{dx}{dt} = 8$, and we want to find $\dfrac{d\theta}{dt}$ when $\ell = 200$. Then we have

$$\tan(\theta) = \frac{y}{x} = \frac{100}{x} \tag{1}$$

so

$$\sec^2(\theta)\frac{d\theta}{dt} = -\frac{100}{x^2}\frac{dx}{dt}. \tag{$\star$}$$

By the Pythagorean theorem, when $\ell = 200$,

$$x = \sqrt{\ell^2 - y^2} = \sqrt{200^2 - 100^2} = 100\sqrt{3}.$$

Then $\sec(\theta) = \frac{\ell}{x} = \frac{200}{100\sqrt{3}} = \frac{2\sqrt{3}}{3}$, so Equation ($\star$) yields

$$\left(\frac{2\sqrt{3}}{3}\right)^2 \frac{d\theta}{dt} = -\frac{100}{(100\sqrt{3})^2}(8) \quad \Longrightarrow \quad \frac{d\theta}{dt} = -\frac{1}{50}.$$

Hence the angle is decreasing at a rate of $\frac{1}{50}$ radians per second.

3. Let the cube's length, width, height and surface area be ℓ, w, h and S, respectively. We are given that $\frac{d\ell}{dt} = \frac{dw}{dt} = \frac{dh}{dt} = -0.5$ and $w = h$, and we want to find $\frac{dS}{dt}$ when $\ell = 8$ and $S = 210$. We know that

$$\begin{aligned} S &= 2\ell w + 2wh + 2\ell h \\ &= 2\ell w + 2w^2 + 2\ell w \\ &= 4\ell w + 2w^2. \end{aligned} \tag{$\dagger$}$$

Differentiation gives

$$\begin{aligned} \frac{dS}{dt} &= 4\ell\frac{dw}{dt} + 4w\frac{d\ell}{dt} + 4w\frac{dw}{dt} \\ &= 4(8)(-0.5) + 4w(-0.5) + 4w(-0.5) \\ &= -16 - 4w. \end{aligned} \tag{$\star$}$$

From Equation (†), when $S = 210$ and $\ell = 8$ we have

$$210 = 4(8)w + 2w^2 \quad \Longrightarrow \quad w = 5 \quad \text{or} \quad w = -21.$$

Width cannot be negative, so it must be that $w = 5$. Hence

$$\frac{dS}{dt} = -16 - 4(5) = -36,$$

that is, the surface area is decreasing at a rate of 36 cm^2/sec.

4. Let x be the distance the man has walked toward the wall and y be the height of his shadow on the wall. Then we know $\frac{dx}{dt} = 1.6$ and we want to find $\frac{dy}{dt}$ when the man is 4 feet from the building, that is, when $x = 8$. This situation is depicted in Figure 1.

From the properties of similar triangles, we know that

$$\begin{aligned} \frac{2}{x} &= \frac{y}{12} \\ xy &= 24 \end{aligned} \tag{$\dagger$}$$

so

$$\frac{dx}{dt}y + \frac{dy}{dt}x = 0. \tag{$\star$}$$

By Equation (†), if $x = 8$ then $y = 3$. Thus, from Equation (⋆),

$$1.6(3) + \frac{dy}{dt}(8) = 0 \quad \Longrightarrow \quad \frac{dy}{dt} = -0.6,$$

so the height of the man's shadow is decreasing at a rate of 0.6 metres per second.

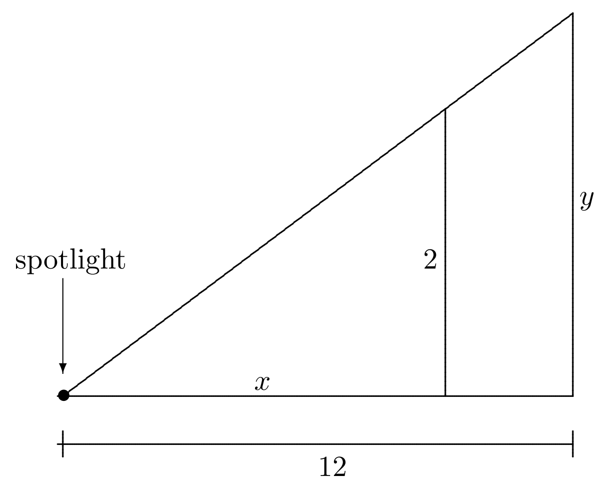

Figure 1: A man walks towards a wall, as in Question 4.

5. Let ℓ be the length of the rope from the bow of the boat to the pulley, h be the height of the pulley above the bow, and x be the distance of the bow from the dock. We are given that $h = 1$ and $\dfrac{d\ell}{dt} = 1$, and we want to determine $\dfrac{dx}{dt}$ when $x = 8$. Then

$$\begin{aligned} x^2 + h^2 &= \ell^2 \\ x^2 + 1 &= \ell^2 \qquad (\dagger) \end{aligned}$$

so

$$\begin{aligned} 2x\frac{dx}{dt} &= 2\ell\frac{d\ell}{dt} \\ x\frac{dx}{dt} &= \ell\frac{d\ell}{dt}. \qquad (\star) \end{aligned}$$

From Equation (†), when $x = 8$,

$$8^2 + 1 = \ell^2 \quad \Longrightarrow \quad \ell^2 = 65 \quad \Longrightarrow \quad \ell \approx 8.06.$$

Hence, from Equation ($\star$), we obtain

$$8\frac{dx}{dt} \approx 8.06(1) \quad \Longrightarrow \quad \frac{dx}{dt} \approx 1.01.$$

The boat is approaching the dock at a speed of about 1.01 metres per second.

6. As shown in Figure 2, let x be the distance of the beam, measured along the coast from the point P directly across from Jack's island, and θ be the angle of the beam measured counterclockwise from the line joining Jack's island (J) to P.

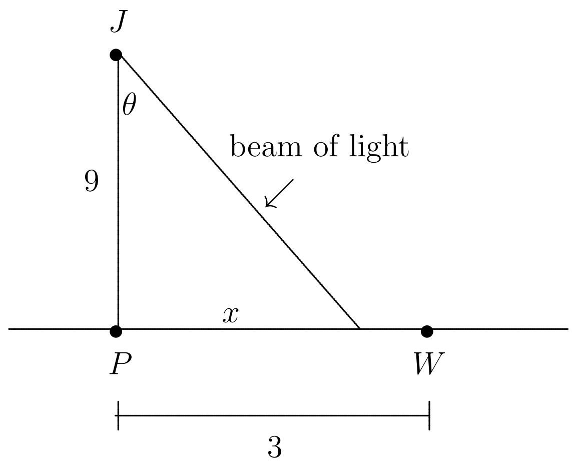

Figure 2: A beam of light moves along the shore, as in Question 6.

Since one revolution is 2π radians, we know that $\frac{d\theta}{dt} = 4(2\pi) = 8\pi$, and we want to determine $\frac{dx}{dt}$ when x is the full distance from P to Will's camp (W), that is, when $x = 3$. We have

$$\tan(\theta) = \frac{x}{9} \tag{$\dagger$}$$

so

$$\sec^2(\theta)\frac{d\theta}{dt} = \frac{1}{9}\frac{dx}{dt}. \tag{$\star$}$$

To determine $\sec(\theta)$, we observe that Equation ($\dagger$) indicates that $\tan(\theta) = \frac{1}{3}$ at the moment of interest. So we are dealing with a right triangle with opposite and adjacent sides of length 1 and 3, respectively, and therefore a hypotenuse of length $\sqrt{10}$ (by the Pythagorean theorem). Thus $\cos(\theta) = \frac{3}{\sqrt{10}}$ and $\sec(\theta) = \frac{\sqrt{10}}{3}$. So, from Equation ($\star$), we have

$$\frac{dx}{dt} = 72\pi\left(\frac{\sqrt{10}}{3}\right)^2 = 80\pi,$$

that is, the beam is travelling along the shoreline at 80π km/min.

7. Let a be the distance Doctor Who has travelled, b be the distance Graham has travelled, and c be the distance between them, as depicted in Figure 3. Then we know that $\frac{da}{dt} = 10$, $\frac{db}{dt} = 6$, and we want to find $\frac{dc}{dt}$.
By the Pythagorean theorem,

$$a^2 + b^2 = c^2 \tag{$\dagger$}$$

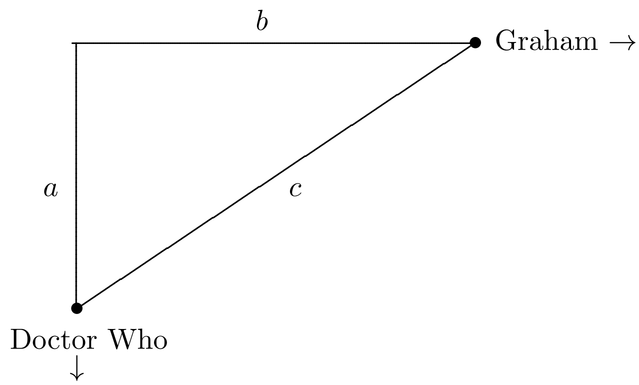

Figure 3: Doctor Who and Graham separating, as in Question 7.

so

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$
$$a\frac{da}{dt} + b\frac{db}{dt} = c\frac{dc}{dt}. \qquad (\star)$$

Five minutes (300 seconds) after the Doctor started, she has travelled $300(10) = 3000$ feet; hence $a = 3000$ at this time. At the same, Graham has only been moving for three minutes (180 seconds) so he has travelled $180(6) = 1080$ feet; hence $b = 1080$. So, by Equation (†), $c \approx 3188.5$. Therefore, from Equation ($\star$),

$$3000(10) + 1080(6) = 3188.5\frac{dc}{dt} \quad \Longrightarrow \quad \frac{dc}{dt} = 11.4,$$

and so the Doctor and Graham separate at about 11.4 feet per second.