MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 3 | **MATHEMATICS 1000-003** | FALL 2022

SOLUTIONS

[5] 1. (a) We use the Chain Rule twice:

$$\begin{aligned} y' &= 3\sin^2(\csc(x)) \cdot [\sin(\csc(x)]' \\ &= 3\sin^2(\csc(x)) \cdot \cos(\csc(x)) \cdot [\csc(x)]' \\ &= 3\sin^2(\csc(x))\cos(\csc(x)) \cdot [-\csc(x)\cot(x)] \\ &= -3\sin^2(\csc(x))\cos(\csc(x))\csc(x)\cot(x). \end{aligned}$$

[5] (b) We use the Chain Rule, followed by the Product Rule:

$$\begin{aligned} y' &= \cos(x^3\csc(x)) \cdot [x^3\csc(x)]' \\ &= \cos(x^3\csc(x)) \cdot ([x^3]'\csc(x) + x^3[\csc(x)]') \\ &= \cos(x^3\csc(x)) \cdot (3x^2\csc(x) + x^3 \cdot [-\csc(x)\cot(x)]) \\ &= \cos(x^3\csc(x)) \cdot (3x^2\csc(x) - x^3\csc(x)\cot(x)) \\ &= 3x^2\csc(x)\cos(x^3\csc(x)) - x^3\csc(x)\cot(x)\cos(x^3\csc(x)). \end{aligned}$$

[5] (c) We use the Product Rule, followed by the Chain Rule:

$$\begin{aligned} y' &= [x^3]'\sin(\csc(x)) + x^3[\sin(\csc(x))]' \\ &= 3x^2\sin(\csc(x)) + x^3 \cdot \cos(\csc(x)) \cdot [\csc(x)]' \\ &= 3x^2\sin(\csc(x)) + x^3\cos(\csc(x)) \cdot [-\csc(x)\cot(x)] \\ &= 3x^2\sin(\csc(x)) - x^3\cos(\csc(x))\csc(x)\cot(x). \end{aligned}$$

[5] (d) Because the function has the form $[f(x)]^{g(x)}$, we need to use logarithmic differentiation:

$$\begin{aligned} \ln(y) &= \ln(x^{\sqrt{x}}) \\ &= \sqrt{x}\ln(x). \end{aligned}$$

To differentiate the righthand side, we need the Product Rule:

$$\begin{aligned}
\frac{d}{dx}[\ln(y)] &= \frac{d}{dx}[\sqrt{x}\ln(x)] \\
\frac{1}{y}\cdot\frac{dy}{dx} &= \frac{d}{dx}[x^{\frac{1}{2}}]\ln(x) + \sqrt{x}\cdot\frac{d}{dx}[\ln(x)] \\
&= \frac{1}{2}x^{-\frac{1}{2}}\cdot\ln(x) + \sqrt{x}\cdot\frac{1}{x} \\
&= \frac{\ln(x)}{2\sqrt{x}} + \frac{1}{\sqrt{x}} \\
&= \frac{\ln(x)+2}{2\sqrt{x}} \\
\frac{dy}{dx} &= y\cdot\frac{\ln(x)+2}{2\sqrt{x}} \\
&= x^{\sqrt{x}}\cdot\frac{\ln(x)+2}{2\sqrt{x}}.
\end{aligned}$$

[5] (e) We use the Product Rule twice:

$$\begin{aligned}
f'(x) &= [x^5]'\ln(x)\tan(x) + x^5[\ln(x)\tan(x)]' \\
&= 5x^4\ln(x)\tan(x) + x^5([\ln(x)]'\tan(x) + \ln(x)[\tan(x)]') \\
&= 5x^4\ln(x)\tan(x) + x^5\left(\frac{1}{x}\cdot\tan(x) + \ln(x)\cdot\sec^2(x)\right) \\
&= 5x^4\ln(x)\tan(x) + x^4\tan(x) + x^5\ln(x)\sec^2(x).
\end{aligned}$$

[5] (f) We use the Quotient Rule, followed by the Chain Rule:

$$\begin{aligned}
f'(x) &= \frac{[e^{3x}-1]'(e^{3x}+1) - (e^{3x}-1)[e^{3x}+1]'}{(e^{3x}+1)^2} \\
&= \frac{e^{3x}\cdot[3x]'\cdot(e^{3x}+1) - (e^{3x}-1)\cdot e^{3x}\cdot[3x]'}{(e^{3x}+1)^2} \\
&= \frac{e^{3x}\cdot 3\cdot(e^{3x}+1) - (e^{3x}-1)\cdot e^{3x}\cdot 3}{(e^{3x}+1)^2} \\
&= \frac{3e^{3x}(e^{3x}+1-e^{3x}+1)}{(e^{3x}+1)^2} \\
&= \frac{3e^{3x}\cdot 2}{(e^{3x}+1)^2} \\
&= \frac{6e^{3x}}{(e^{3x}+1)^2}.
\end{aligned}$$

[5] 2. We use implicit differentiation, applying the Product Rule to the lefthand side:

$$\begin{aligned}
\frac{d}{dx}[x^2\cos(y)] &= \frac{d}{dx}[\sec(6x) - 8y] \\
\frac{d}{dx}[x^2]\cos(y) + x^2\cdot\frac{d}{dx}[\cos(y)] &= \sec(6x)\tan(6x)\cdot\frac{d}{dx}[6x] - 8\frac{dy}{dx} \\
2x\cos(y) + x^2\cdot\left[-\sin(y)\frac{dy}{dx}\right] &= \sec(6x)\tan(6x)\cdot 6 - 8\frac{dy}{dx} \\
8\frac{dy}{dx} - x^2\sin(y)\frac{dy}{dx} &= 6\sec(6x)\tan(6x) - 2x\cos(y) \\
\frac{dy}{dx}[8 - x^2\sin(y)] &= 6\sec(6x)\tan(6x) - 2x\cos(y) \\
\frac{dy}{dx} &= \frac{6\sec(6x)\tan(6x) - 2x\cos(y)}{8 - x^2\sin(y)}.
\end{aligned}$$

[5] 3. Let $g(x) = k\cdot f(x)$ so $g(x+h) = k\cdot f(x+h)$. Then

$$\begin{aligned}
[k\cdot f(x)]' = g'(x) &= \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0}\frac{k\cdot f(x+h) - k\cdot f(x)}{h} \\
&= \lim_{h\to 0}\frac{k[f(x+h) - f(x)]}{h} \\
&= k\cdot\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\
&= k\cdot f'(x).
\end{aligned}$$