MEMORIAL UNIVERSITY OF NEWFOUNDLAND

DEPARTMENT OF MATHEMATICS AND STATISTICS

TEST 2 — **MATHEMATICS 1000-002** — FALL 2022

SOLUTIONS

[3] 1. (a) A function may have no more than two horizontal asymptotes. The horizontal asymptotes describe the behaviour of the function as $\to \pm\infty$, and thus the graph of $f(x)$ may cross a horizontal asymptote any number of times for intermediate values of x. It's a vertical asymptote, not a horizontal asymptote, that describes the manner in which $f(x)$ becomes unboundedly large as x approaches a real number p. Hence the correct choice is:

☐ a horizontal asymptote describes the behaviour of $f(x)$ as x becomes unboundedly large (positively or negatively)

[3] (b) By definition, the only which ensures the existence of a removable discontinuity is:

☐ $\lim\limits_{x \to p} f(x)$ exists, but $f(p)$ is undefined

[3] (c) We proved that differentiability implies continuity, so the following is impossibe:

☐ $f(x)$ is differentiable at $x = p$ but not continuous at $x = p$

[3] (d) Graphically, a function is non-differentiable at a point if it has a vertical tangent line at a point (because this implies that the one-sided limits of the definition of the derivative are infinite), an abrupt change or "sharp corner" at a point (because this implies that the one-sided limits of the definition of the derivative are not equal), or if it has a vertical tangent asymptote at a point (because this implies that the function is not continuous there). Hence the correct choice is:

☐ $f(x)$ has a horizontal tangent line at $x = p$

[3] (e) The slope of a tangent line, the velocity of an object, and the infection rate of a virus are all examples of rates of change. Furthermore, the derivative is the mathematical equivalent of a rate of change. Hence the correct choice is:

☐ all of the above are examples of, or are equivalent to, a rate of change

[5] 2. Observe that $f(x)$ is a rational function, so we need only consider one of the limits at infinity.

Then

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2x^3(6x-1)}{(3x^2+4)^2} \\
&= \lim_{x\to\infty} \frac{12x^4-2x^3}{9x^4+24x^2+16} \cdot \frac{\frac{1}{x^4}}{\frac{1}{x^4}} \\
&= \lim_{x\to\infty} \frac{12-\frac{2}{x}}{9+\frac{24}{x^2}+\frac{16}{x^4}} \\
&= \frac{12-0}{9+0+0} \\
&= \frac{12}{9} \\
&= \frac{4}{3}.
\end{aligned}$$

Hence the only horizontal asymptote is the line $y = \frac{4}{3}$.

[8] 3. First we observe that $f(2) = 3k^2 - 4$, which is defined for all k. Next we need to determine if $\lim_{x\to 2} f(x)$ exists. Since the definition of $f(x)$ changes at $x = 2$, we consider the one-sided limits. From the left we have

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 + 5kx) = 4 + 10k,$$

and from the right we have

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (k^2x + 4x + 4) = 2k^2 + 8 + 4 = 2k^2 + 12.$$

We set

$$\begin{aligned}
\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^+} f(x) \\
4 + 10k &= 2k^2 + 12 \\
2k^2 - 10k + 8 &= 0 \\
k^2 - 5k + 4 &= 0 \\
(k-4)(k-1) &= 0.
\end{aligned}$$

Thus $k = 4$ or $k = 1$.

For $k = 4$, we have $f(2) = 44$ and $\lim_{x\to 2} f(x) = 44$, so $f(x)$ is continuous.

For $k = 1$, we have $f(2) = -1$ and $\lim_{x\to 2} f(x) = 14$, so $f(x)$ is not continuous.

Hence the only value of k for which $f(x)$ is continuous at $x = 2$ is $k = 4$.

[8] 4. (a) We have

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{2(x+h)+5} - \frac{x}{2x+5}}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(2x+5) - x(2x+2h+5)}{(2x+5)(2x+2h+5)} \\
&= \lim_{h\to 0} \frac{2x^2+2xh+5x+5h-2x^2-2xh-5x}{h(2x+5)(2x+2h+5)} \\
&= \lim_{h\to 0} \frac{5h}{h(2x+5)(2x+2h+5)} \\
&= \lim_{h\to 0} \frac{5}{(2x+5)(2x+2h+5)} \\
&= \frac{5}{(2x+5)(2x+5)} \\
&= \frac{5}{(2x+5)^2}.
\end{aligned}$$

[4] (b) From part (a), $m = f'(-3) = 5$. Furthermore, $y = f(-3) = 3$. Thus the equation of the tangent line is

$$y - 3 = 5(x+3) \quad \Longrightarrow \quad y = 5x + 18.$$