

MEMORIAL UNIVERSITY OF NEWFOUNDLAND
DEPARTMENT OF MATHEMATICS AND STATISTICS

SECTION 2.2

Math 1000 Worksheet

FALL 2022

SOLUTIONS

$$\begin{aligned}
 1. \text{ (a) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{3(x+h)}{x+h-4} - \frac{3x}{x-4}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3(x+h)(x-4) - 3x(x+h-4)}{h(x-4)(x+h-4)} = \lim_{h \rightarrow 0} \frac{-12h}{h(x-4)(x+h-4)} \\
 &= \lim_{h \rightarrow 0} \frac{-12}{(x-4)(x+h-4)} = -\frac{12}{(x-4)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b) } f'(t) &= \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} = \lim_{h \rightarrow 0} \frac{\frac{4}{(t+h)^2+1} - \frac{4}{t^2+1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{4(t^2+1) - 4[(t+h)^2+1]}{h(t^2+1)[(t+h)^2+1]} = \lim_{h \rightarrow 0} \frac{-8th - 4h^2}{h(t^2+1)[(t+h)^2+1]} \\
 &= \lim_{h \rightarrow 0} \frac{-4h(h+2t)}{h(t^2+1)[(t+h)^2+1]} = \lim_{h \rightarrow 0} \frac{-4(h+2t)}{(t^2+1)[(t+h)^2+1]} = -\frac{8t}{(t^2+1)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c) } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{2-3(x+h)} - \sqrt{2-3x}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{2-3(x+h)} - \sqrt{2-3x}}{h} \cdot \frac{\sqrt{2-3(x+h)} + \sqrt{2-3x}}{\sqrt{2-3(x+h)} + \sqrt{2-3x}} \\
 &= \lim_{h \rightarrow 0} \frac{[2-3(x+h)] - [2-3x]}{h[\sqrt{2-3(x+h)} + \sqrt{2-3x}]} \\
 &= \lim_{h \rightarrow 0} \frac{-3h}{h[\sqrt{2-3(x+h)} + \sqrt{2-3x}]} = \lim_{h \rightarrow 0} \frac{-3}{\sqrt{2-3(x+h)} + \sqrt{2-3x}} \\
 &= -\frac{3}{2\sqrt{2-3x}}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d) } g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sqrt{(x+h)^2+1} - \sqrt{x^2+1}}{h} \cdot \frac{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^2+1] - [x^2+1]}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})} \\
 &= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h(\sqrt{(x+h)^2+1} + \sqrt{x^2+1})} = \lim_{h \rightarrow 0} \frac{2x+h}{\sqrt{(x+h)^2+1} + \sqrt{x^2+1}} \\
 &= \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}
 \end{aligned}$$

2. The derivative of $f(x)$ is

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 3(x+h)^2 + 2(x+h)] - [x^3 - 3x^2 + 2x]}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3x^2h + 3xh^2 + h^3 - 6xh - 3h^2 + 2h}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h(3x^2 + 3xh + h^2 - 6x - 3h + 2)}{h} \\
 &= \lim_{h \rightarrow 0} (3x^2 + 3xh + h^2 - 6x - 3h + 2) \\
 &= 3x^2 - 6x + 2.
 \end{aligned}$$

At $x = 3$, $f'(3) = 11$ so the equation of the tangent line has the form $y = 11x + b$. Since $f(3) = 6$, we know that

$$6 = 11(3) + b \implies b = -27$$

and so the equation of the tangent line is $y = 11x - 27$. Finally, note that $f'(-1) = 11$ as well, so since the tangent lines to $f(x)$ at $x = 3$ and at $x = -1$ have the same slope, they must be parallel.

3. First note that

$$|3x + 6| = \begin{cases} 3x + 6 & \text{for } x \geq -2 \\ -(3x + 6) & \text{for } x < -2. \end{cases}$$

Using the alternative definition of the derivative,

$$\lim_{x \rightarrow -2^-} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2^-} \frac{-3x - 6 - 0}{x + 2} = \lim_{x \rightarrow -2^-} \frac{-3(x + 2)}{x + 2} = \lim_{x \rightarrow -2^-} (-3) = -3,$$

whereas

$$\lim_{x \rightarrow -2^+} \frac{f(x) - f(-2)}{x + 2} = \lim_{x \rightarrow -2^+} \frac{(3x + 6) - 0}{x + 2} = \lim_{x \rightarrow -2^+} \frac{3(x + 2)}{x + 2} = \lim_{x \rightarrow -2^+} 3 = 3.$$

Since the one-sided limits are not equal,

$$f'(-2) = \lim_{x \rightarrow -2} \frac{f(x) - f(-2)}{x + 2}$$

does not exist, and hence $f(x)$ is non-differentiable at $x = -2$.

4. By the alternative definition of the derivative,

$$f'(-4) = \lim_{x \rightarrow -4} \frac{f(x) - f(-4)}{x - (-4)} = \lim_{x \rightarrow -4} \frac{f(x) + 31}{x + 4}.$$

From the left,

$$\begin{aligned}\lim_{x \rightarrow -4^-} \frac{f(x) + 31}{x + 4} &= \lim_{x \rightarrow -4^-} \frac{(17 - 3x^2) + 31}{x + 4} = \lim_{x \rightarrow -4^-} \frac{48 - 3x^2}{x + 4} \\ &= \lim_{x \rightarrow -4^-} \frac{-3(x - 4)(x + 4)}{x + 4} = \lim_{x \rightarrow -4^-} -3(x - 4) = 24.\end{aligned}$$

From the right,

$$\begin{aligned}\lim_{x \rightarrow -4^+} \frac{f(x) + 31}{x + 4} &= \lim_{x \rightarrow -4^+} \frac{(x^3 + 3x^2 - 15) + 31}{x + 4} = \lim_{x \rightarrow -4^+} \frac{x^3 + 3x^2 + 16}{x + 4} \\ &= \lim_{x \rightarrow -4^+} \frac{(x + 4)(x^2 - x + 4)}{x + 4} = \lim_{x \rightarrow -4^+} (x^2 - x + 4) = 24.\end{aligned}$$

Since the one-sided limits agree, the limit exists and therefore $f(x)$ is differentiable at $x = -4$.