

# MEMORIAL UNIVERSITY OF NEWFOUNDLAND

## DEPARTMENT OF MATHEMATICS AND STATISTICS

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SECTION 1.7

Math 1000 Worksheet

FALL 2022

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### SOLUTIONS

1. (a) The only value of  $x$  for which either definition of  $f(x)$  is undefined is at  $x = 2$ , which is also the point at which the function definition changes. Here,  $f(2) = 0$ . We do not need to evaluate the one-sided limits because  $f(x)$  is defined in the same way to either side of  $x = 2$ . We simply have

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 4.$$

However, this means that  $f(2) \neq \lim_{x \rightarrow 2} f(x)$ . Thus  $x = 2$  is a removable discontinuity.

- (b) First we identify any points at which the function will not be defined. This can only happen when the first part of the definition has a zero denominator, and since

$$\frac{x + 1}{x^2 - x - 2} = \frac{x + 1}{(x - 2)(x + 1)}$$

this occurs for  $x = 2$  and  $x = -1$ . However,  $f(x)$  only adopts this definition for  $x < 1$ , so only  $x = -1$  is a discontinuity. To classify it, we need to take the limit at  $x \rightarrow -1$ . Since the numerator is also zero when  $x = -1$ , we have a  $\frac{0}{0}$  indeterminate form. Hence we use the cancellation method:

$$\lim_{x \rightarrow -1} \frac{x + 1}{x^2 - x - 2} = \lim_{x \rightarrow -1} \frac{x + 1}{(x - 2)(x + 1)} = \lim_{x \rightarrow -1} \frac{1}{x - 2} = -\frac{1}{3}.$$

Since the limit exists,  $x = -1$  is a removable discontinuity.

We also need to check the points where the definition of  $f(x)$  changes; the only such point is  $x = 1$ . We have  $f(1) = 2$ . Also,

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{x + 1}{x^2 - x - 2} = \lim_{x \rightarrow 1^-} \frac{x + 1}{(x - 2)(x + 1)} = \lim_{x \rightarrow 1^-} \frac{1}{x - 2} = -1$$

and

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (3 - x^2) = 2.$$

Hence the limit as  $x \rightarrow 1$  does not exist, and so  $x = 1$  is a discontinuity. Since the one-sided limits both exist, however,  $x = 1$  is a jump discontinuity.

(c) First, note that  $\frac{x}{x^2 - 5x}$  is undefined if

$$x^2 - 5x = x(x - 5) = 0.$$

This occurs when  $x = 0$  or  $x = 5$ , but we ignore the second possibility because this definition only applies for  $x < 1$ . At  $x = 0$ , direct substitution gives a  $\frac{0}{0}$  indeterminate form, so we take the limit using the cancellation method:

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{x}{x^2 - 5x} = \lim_{x \rightarrow 0} \frac{x}{x(x - 5)} = \lim_{x \rightarrow 0} \frac{1}{x - 5} = \frac{1}{0 - 5} = -\frac{1}{5}.$$

Since the limit exists, there is **an removable discontinuity at  $x = 0$ .**

Next, observe that  $\frac{2}{x - 9}$  is undefined if  $x - 9 = 0$ , that is, for  $x = 9$ . Direct substitution produces a  $\frac{K}{0}$  form, so this is a vertical asymptote and we can conclude that  $\lim_{x \rightarrow 9} f(x)$  does not exist. Thus there is **an infinite discontinuity at  $x = 9$ .**

Lastly, we must consider  $x = 1$ , since this is where the definition of the piecewise function changes. First, note that

$$f(1) = \frac{2}{1 - 9} = -\frac{1}{4}.$$

Because  $f(x)$  is defined differently to the left and to the right of  $x = 1$ , we evaluate the one-sided limits:

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \frac{x}{x^2 - 5x} = \frac{1}{1 - 5} = -\frac{1}{4}$$

and

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} \frac{2}{x - 9} = \frac{2}{1 - 9} = -\frac{1}{4}.$$

Hence

$$\lim_{x \rightarrow 1} f(x) = -\frac{1}{4} = f(1),$$

and so  $x = 1$  is not a discontinuity at all.

2. Observe that  $f(x)$  is a continuous function (since it is a polynomial), and  $f(-2) = 59$  while  $f(2) = -21$ . By the Intermediate Value Theorem, there must be at least one  $x$  on the interval  $[-2, 2]$  for which  $f(x) = 0$ , which means that there **must be a root** of  $f(x)$  on that interval.