Please remember that each final exam is different, and the style and emphasis of the questions will vary from semester to semester.

1. Evaluate each limit or explain why it does not exist. Assign ∞ or $-\infty$ when appropriate. You may not use l'Hôpital's Rule.

[3] (a) $\lim_{x\to 5} \frac{25 - x^2}{x^2 - 2x - 15}$

[3] (b) $\lim_{x\to -1} \frac{3x}{(x+1)^2}$

[3] (c) $\lim_{x\to 0} \frac{x}{1 - \sqrt{1+x}}$

2. Consider the function

$$f(x) = \begin{cases} \dfrac{x^2-4}{x^2-x-2}, & \text{for } x \geq 0 \\ \dfrac{4}{x+2}, & \text{for } x < 0. \end{cases}$$

You may not use l'Hôpital's Rule for either part of this question.

[3] (a) Use the definition of continuity to determine whether $f(x)$ is continuous at $x = 0$. If it is not, is the discontinuity removable or non-removable?

[5] (b) Use the definition of continuity to determine all other points at which $f(x)$ is not continuous. Classify any discontinuities as removable or non-removable.

3. Consider the function $f(x) = \sqrt{2x+1}$.

[7] (a) Use the definition of derivative to find $f'(x)$.

[3] (b) Find the equation of the tangent line to the graph of $y = f(x)$ at $x = 4$.

4. Find the derivative of y with respect to x, making any obvious simplifications.

[3] (a) $y = x^4 e^{x^3}$

[4] (b) $y = \cos\left(\sqrt{x}\right) + \sqrt{\cosh(x)}$

[5] (c) $y - \frac{x}{y} = \tan(y)$

[5] (d) $y = \frac{(x^2+1)^x}{x}$

[4] 5. (a) Find $y'(1)$ given $y = \frac{\arctan(x)}{x^2+1}$.

[2] (b) Determine, with justification, the 50th derivative of $f(x) = x^2(9x^5-7)^3(x^8+3x^5+1)^4$.

[7] 6. An airplane flying at an altitude of 6 km passed directly over a radar antenna. When the distance from the radar antenna to the plane was 10 km, the radar detected that the distance to the airplane was changing at a rate of 200 km per hour. What was the speed of the plane?

7. Consider the function $f(x) = \dfrac{4x(x+3)}{(x+1)^2}$ with derivatives

$$f'(x) = \frac{4(3-x)}{(x+1)^3} \quad \text{and} \quad f''(x) = \frac{8(x-5)}{(x+1)^4}.$$

[3] (a) Find the points of discontinuity, including the vertical asymptotes, of the graph of $f(x)$, if any.

[3] (b) Find the horizontal asymptotes of the graph of $f(x)$, if any.

[2] (c) Find the x- and y-intercepts of the graph of $f(x)$, if any.

[3] (d) Determine the intervals on which $f(x)$ is increasing or decreasing, and classify any relative (local) extrema.

[3] (e) Determine the intervals on which $f(x)$ is concave upward or concave downward, and identify any points of inflection.

[3] (f) Sketch the graph of $f(x)$. Label your graph carefully.

[7] 8. A rectangular billboard is being constructed with an area of 81 square metres. However, any display erected on the billboard must have blank margins of width 1 metre on each side and 4 metres on the top and bottom. Find the dimensions of the billboard which will ensure that the display area is as large as possible.

9. Use l'Hôpital's Rule to evaluate each of the following.

[4] (a) $\displaystyle\lim_{x\to 0} \frac{e^x - x - 1}{x^2 - x + \sin(x)}$

[5] (b) $\displaystyle\lim_{x\to\infty} \left(\frac{x}{x+1}\right)^x$

[6] 10. Use the definition of the derivative to prove the Constant Multiple Rule, that is, that if $f(x)$ is a differentiable function then

$$[kf(x)]' = kf'(x)$$

for any constant k.

[4] 11. Give a piecewise function that is defined everywhere, except at $x = -2$ and $x = 4$, which are also the points where its behaviour changes. The limits from the left and right should exist as x approaches both -2 and 4, but the limit should exist only as x approaches -2.