The quasi-isometry relation for finitely generated groups

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Cayley graphs of finitely generated groups

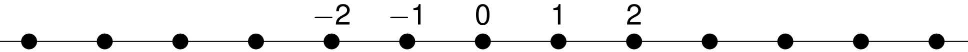
Definition

Let G be a f.g. group and let $S \subseteq G \setminus \{1_G\}$ be a finite generating set. Then the Cayley graph $\mathrm{Cay}(G, S)$ *is the graph with vertex set G and edge set*

$$E = \{\{x, y\} \mid y = xs \text{ for some } s \in S \cup S^{-1}\}.$$

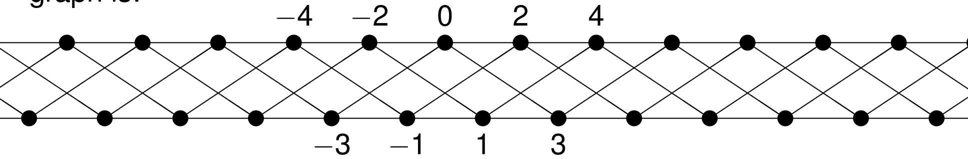
The corresponding word metric *is denoted by d_S.*

For example, when $G = \mathbb{Z}$ and $S = \{1\}$, then the corresponding Cayley graph is:



But which Cayley graph?

However, when $G = \mathbb{Z}$ and $S = \{2, 3\}$, then the corresponding Cayley graph is:



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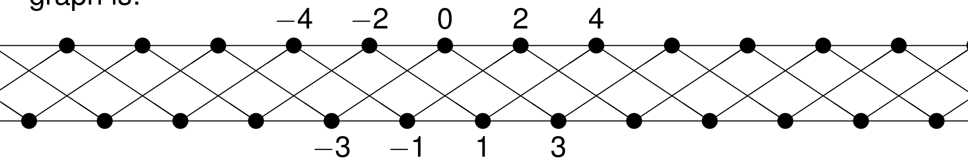
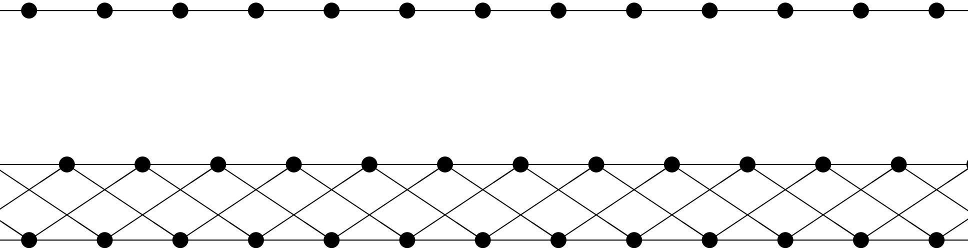


Theorem (S.T.)

There does not exist an explicit choice of generators for each f.g. group which has the property that isomorphic groups are assigned isomorphic Cayley graphs.

The basic idea of geometric group theory

Although the Cayley graphs of a f.g. group G with respect to different generating sets S are usually nonisomorphic, they always have the same large scale geometry.



The quasi-isometry relation

Definition (Gromov)

Let G, H be f.g. groups with word metrics d_S, d_T respectively. Then G, H are said to be quasi-isometric, written $G \approx_{QI} H$, iff there exist

- *constants $\lambda \geq 1$ and $C \geq 0$, and*
- *a map $\varphi : G \to H$*

such that for all $x, y \in G$,

$$\frac{1}{\lambda} d_S(x, y) - C \leq d_T(\varphi(x), \varphi(y)) \leq \lambda d_S(x, y) + C;$$

and for all $z \in H$,

$$d_T(z, \varphi[G]) \leq C.$$

When $C = 0$

Definition (Gromov)

Let G, H be f.g. groups with word metrics d_S, d_T respectively. Then G, H are said to be Lipschitz equivalent iff there exist

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As expected ...

Observation

If S, S' are finite generating sets for G, then

$$id : \langle G, d_S \rangle \longrightarrow \langle G, d_{S'} \rangle$$

is a quasi-isometry.

Thus while it doesn't make sense to talk about the isomorphism type of "the Cayley graph of G", it does make sense to talk about the quasi-isometry type.

A topological criterion

Theorem (Gromov)

If G, H are f.g. groups, then the following are equivalent.

- *G and H are quasi-isometric.*
- *There exists a locally compact space X on which G, H have commuting proper actions via homeomorphisms such that X/G and X/H are both compact.*

Definition

The action of the discrete group G on X is proper iff for every compact subset $K \subseteq X$, the set $\{g \in G \mid g(K) \cap K \neq \emptyset\}$ is finite.

Obviously quasi-isometric groups

Definition

Two groups G_1, G_2 are said to be virtually isomorphic, written $G_1 \approx_{VI} G_2$, iff there exist subgroups $N_i \leqslant H_i \leqslant G_i$ such that:

- $[G_1 : H_1], [G_2 : H_2] < \infty$.
- *N_1, N_2 are finite normal subgroups of H_1, H_2 respectively.*
- $H_1/N_1 \cong H_2/N_2$.

Proposition (Folklore)

If the f.g. groups G_1, G_2 are virtually isomorphic, then G_1, G_2 are quasi-isometric.

More quasi-isometric groups

Theorem (Erschler)

The f.g. groups Alt(5) *wr* $\mathbb{Z}$ *and* C_{60} *wr* $\mathbb{Z}$ *are quasi-isometric but not virtually isomorphic. (In fact, they have isomorphic Cayley graphs.)*

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Question

How many f.g. groups up to quasi-isometry?

Growth rates and quasi-isometric groups

Theorem (Grigorchuk 1984 - Bowditch 1998)

There are $2^{\aleph_0}$ f.g. groups up to quasi-isometry.

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Proof (Grigorchuk).

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Proof (Grigorchuk).

Consider the growth rate of the size of balls of radius n in the Cayley graphs of suitably chosen groups. □

Proof (Bowditch).

Consider the growth rate of the length of “irreducible loops” in the Cayley graphs of suitably chosen groups. □

The complexity of the quasi-isometry relation

Question

What are the possible complete invariants for the quasi-isometry problem for f.g. groups?

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What are the possible complete invariants for the quasi-isometry problem for f.g. groups?

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Is the quasi-isometry problem for f.g. groups strictly harder than the isomorphism problem?

An explicit reduction

Let S be a fixed infinite f.g. simple group. Then the isomorphism problem for f.g. groups can be reduced to the virtual isomorphism problem via the explicit map

$$G \mapsto (\mathrm{Alt}(5) \text{ wr } G) \text{ wr } S$$

in the sense that

$$G \cong H \quad \text{iff} \quad (\mathrm{Alt}(5) \text{ wr } G) \text{ wr } S \approx_{VI} (\mathrm{Alt}(5) \text{ wr } H) \text{ wr } S.$$

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Church's Thesis for Real Mathematics

EXPLICIT = BOREL

A function $f : X \to Y$ is Borel iff graph(f) is a Borel subset of $X \times Y$. "Equivalently", $f^{-1}(A)$ is Borel for each Borel subset $A \subseteq Y$.

The Polish space of f.g. groups

Let $\mathbb{F}_m$ be the free group on $\{x_1, \cdots, x_m\}$ and let $\mathcal{G}_m$ be the compact space of normal subgroups of $\mathbb{F}_m$. Since each m-generator group can be realised as a quotient $\mathbb{F}_m/N$ for some $N \in \mathcal{G}_m$, we can regard $\mathcal{G}_m$ as the space of m-generator groups. There are natural embeddings

$$\mathcal{G}_1 \hookrightarrow \mathcal{G}_2 \hookrightarrow \cdots \hookrightarrow \mathcal{G}_m \hookrightarrow \cdots$$

and we can regard

$$\mathcal{G} = \bigcup_{m \geq 1} \mathcal{G}_m$$

as the space of f.g. groups.

A slight digression

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Some Isolated Points

- Finite groups
- Finitely presented simple groups

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The Next Stage

- $SL_3(\mathbb{Z})$

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Question (Grigorchuk)

What is the Cantor-Bendixson rank of $\mathcal{G}_m$?

Borel equivalence relations

Remark (Champetier)

The isomorphism relation $\cong$ on the space $\mathcal{G}$ of f.g. groups is a countable Borel equivalence relation.

Definition

- *An equivalence relation E on a Polish space X is Borel iff E is a Borel subset of $X \times X$.*
- *A Borel equivalence relation E is countable iff every E-class is countable.*

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Theorem (Feldman-Moore)

Every countable Borel equivalence relation can be realized as the orbit equivalence relation of a Borel action of a countable group.

The isomorphism relation

The natural action of the countable group $\text{Aut}(\mathbb{F}_m)$ on $\mathbb{F}_m$ induces a corresponding homeomorphic action on the compact space $\mathcal{G}_m$ of normal subgroups of $\mathbb{F}_m$. Furthermore, each $\pi \in \text{Aut}(\mathbb{F}_m)$ extends to a homeomorphism of the space $\mathcal{G}$ of f.g. groups.

If $N, M \in \mathcal{G}_m$ and there exists $\pi \in \text{Aut}(\mathbb{F}_m)$ such that $\pi(N) = M$, then $\mathbb{F}_m/N \cong \mathbb{F}_m/M$. Unfortunately, the converse does not hold.

The isomorphism relation continued

Theorem (Tietze)

If $N, M \in \mathcal{G}_m$, then the following are equivalent:

- $\mathbb{F}_m / N \cong \mathbb{F}_m / M$.
- *There exists* $\pi \in \mathrm{Aut}(\mathbb{F}_{2m})$ *such that* $\pi(N) = M$.

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Corollary (Champetier)

The isomorphism relation $\cong$ on the space $\mathcal{G}$ of f.g. groups is the orbit equivalence relation arising from the homeomorphic action of the countable group $\mathrm{Aut}_f(\mathbb{F}_\infty)$ of finitary automorphisms of the free group $\mathbb{F}_\infty$ on $\{x_1, x_2, \cdots, x_m, \cdots\}$.

Some Borel equivalence relations

Remark

The following are Borel equivalence relations on the space $\mathcal{G}$ of f.g. groups:

- the isomorphism relation $\cong$
- the virtual isomorphism relation $\approx_{VI}$
- the quasi-isometry relation $\approx_{QI}$

Borel reductions

Definition

Let E, F be Borel equivalence relations on the Polish spaces X, Y.

- $E \leq_B F$ *iff there exists a Borel map* $f : X \to Y$ *such that*

$$x \, E \, y \iff f(x) \, F \, f(y).$$

In this case, f is called a **Borel reduction** *from E to F.*

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- $E \sim_B F$ *iff both* $E \leq_B F$ *and* $F \leq_B E$*.*

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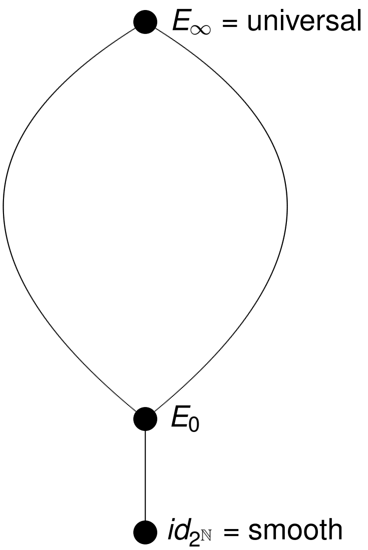
Countable Borel equivalence relations



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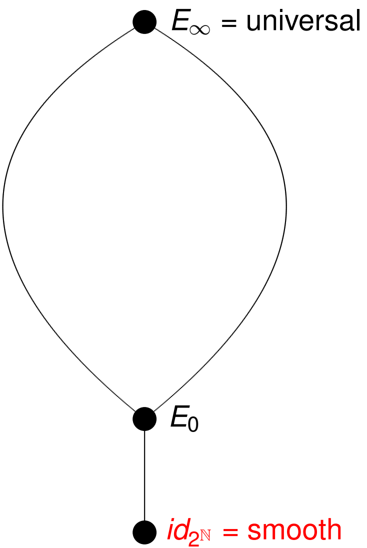

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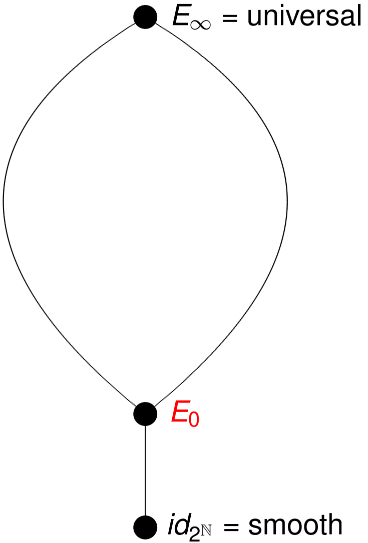

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E_0 is the equivalence relation of eventual equality on the space $2^{\mathbb{N}}$ of infinite binary sequences.

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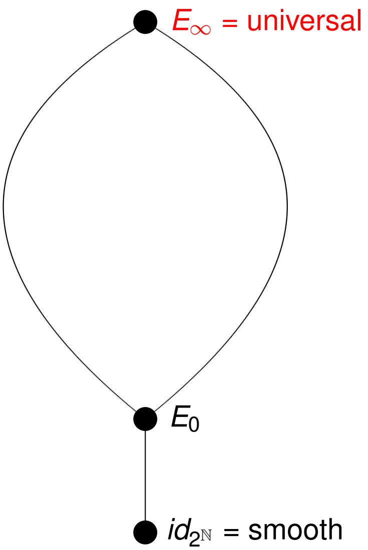

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A countable Borel equivalence relation E is universal iff $F \leq_B E$ for every countable Borel equivalence relation F.

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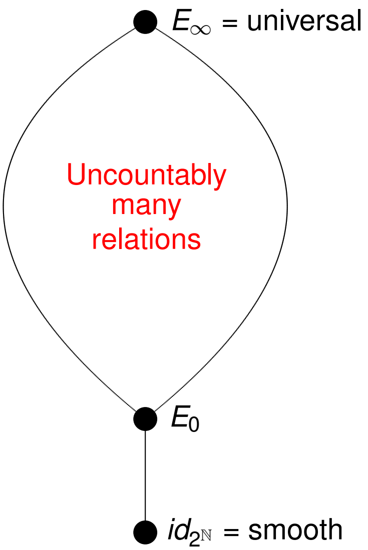

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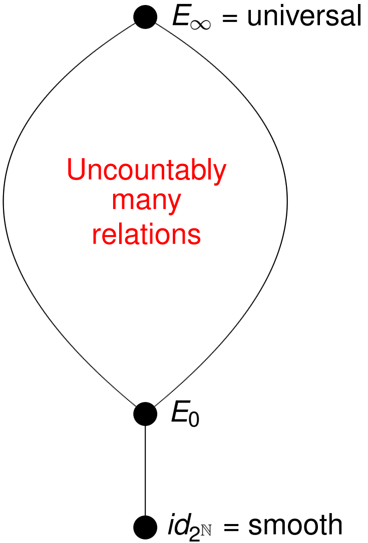

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Question

Where does $\cong$ fit in?

A universal countable Borel equivalence relation

Confirming a conjecture of Hjorth-Kechris ...

Theorem (S.T.-Velickovic)

The isomorphism relation $\cong$ on the space $\mathcal{G}$ of f.g. groups is a universal countable Borel equivalence relation.

Remark

The proof shows that the isomorphism relation on the space $\mathcal{G}_5$ of 5-generator groups is already countable universal. Presumably the same is true for the isomorphism relation on $\mathcal{G}_2$?

The commensurability relation $\approx_C$

Definition

The f.g. groups G_1, G_2 are (abstractly) commensurable, written $G_1 \approx_C G_2$, iff there exist subgroups $H_i \leqslant G_i$ of finite index such that $H_1 \cong H_2$.

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Observation

The commensurability relation $\approx_C$ on the space $\mathcal{G}$ of f.g. groups is a countable Borel equivalence relation.

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Open Problem

Find a "group-theoretic" reduction from $\approx_C$ to $\cong$.

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Open Problem

Find a "group-theoretic" reduction from $\approx_C$ to $\cong$.

Theorem (S.T.)

There does not exist a Borel reduction f from $\approx_C$ to $\cong$ such that $f(G) \approx_C G$ for all $G \in \mathcal{G}$.

The virtual isomorphism relation

Definition

The f.g. groups G_1, G_2 are virtually isomorphic, written $G_1 \approx_V G_2$, iff there exist subgroups $N_i \leqslant H_i \leqslant G_i$ such that:

- $[G_1 : H_1], [G_2 : H_2] < \infty$.
- *N_1, N_2 are finite normal subgroups of H_1, H_2 respectively.*
- $H_1/N_1 \cong H_2/N_2$.

Theorem (S.T.)

The virtual isomorphism problem for f.g. groups is strictly harder than the isomorphism problem.

Definition

E_1 is the Borel equivalence relation on $[0,1]^{\mathbb{N}}$ defined by

$$x \; E_1 \; y \iff x(n) = y(n) \text{ for almost all } n.$$

Theorem (Kechris-Louveau)

E_1 is not Borel reducible to the isomorphism relation on any class of countable structures.

Central Extensions of Tarski Monsters

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Lemma (S.T.)

There exists a Borel map $s \mapsto G_s$ from $[0,1]^{\mathbb{N}}$ to $\mathcal{G}$ such that:

- *G_s is a suitable central extension of a fixed Tarski monster M.*
- *$s \, E_1 \, t$ iff $G_s \approx_{VI} G_t$.*

$\mathbf{K}_\sigma$ equivalence relations

Definition

The equivalence relation E on the Polish space X is $\mathbf{K}_\sigma$ iff E is the union of countably many compact subsets of $X \times X$.

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Example

The following are $\mathbf{K}_\sigma$ equivalence relations on the space $\mathcal{G}$ of f.g. groups:

- the isomorphism relation $\cong$
- the virtual isomorphism relation $\approx_{VI}$
- the quasi-isometry relation $\approx_{QI}$

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The quasi-isometry relation is $\mathbf{K}_\sigma$

- Fix some $m \geq 2$.
- Let $G, H \in \mathcal{G}_m$ with word metrics d_S, d_T respectively.
- Suppose that there exists a (λ, C)-quasi-isometry $\varphi : G \to H$.

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- Clearly we can suppose that $\varphi(1_G) = 1_H$.
- Then for every $g \in G$, there are only finitely many possibilities for $\varphi(g) \in H$.
- And for every $h \in H$, there are only finitely many possibilities for $g \in G$ such that $d_T(h, \varphi(g)) \leq C$.

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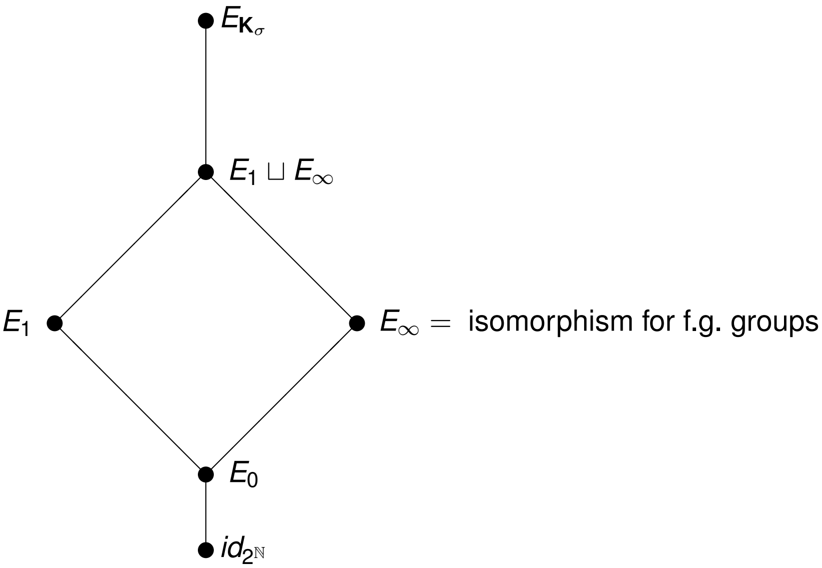
- Clearly we can suppose that $\varphi(1_G) = 1_H$.
- Then for every $g \in G$, there are only finitely many possibilities for $\varphi(g) \in H$.
- And for every $h \in H$, there are only finitely many possibilities for $g \in G$ such that $d_T(h, \varphi(g)) \leq C$.
- Thus the relation

$$E_{\lambda,C} = \{(G, H) \mid G, H \text{ are } (\lambda, C)\text{-quasi-isometric}\}$$

is a compact subset of $\mathcal{G}_m \times \mathcal{G}_m$.

$\mathbf{K}_\sigma$ equivalence relations



Some universal $\mathbf{K}_\sigma$ equivalence relations

Theorem (Rosendal)

Let $E_{\mathbf{K}_\sigma}$ be the equivalence relation on $\prod_{n\geq 1}\{1,\dots,n\}$ defined by

$$\alpha \; E_{\mathbf{K}_\sigma} \; \beta \iff \exists N \, \forall k \;\; |\alpha(k) - \beta(k)| \leq N.$$

Then $E_{\mathbf{K}_\sigma}$ is a universal $\mathbf{K}_\sigma$ equivalence relation.

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Theorem (Rosendal)

The Lipschitz equivalence relation on the space of compact separable metric spaces is Borel bireducible with $E_{\mathbf{K}_\sigma}$.

More universal $\mathbf{K}_\sigma$ equivalence relations

Theorem (S.T.)

The following equivalence relations are Borel bireducible with $E_{\mathbf{K}_\sigma}$

- *the growth rate relation on the space of strictly increasing functions $f : \mathbb{N} \to \mathbb{N}$;*
- *the quasi-isometry relation on the space of connected 4-regular graphs.*

Definition

The strictly increasing functions $f, g : \mathbb{N} \to \mathbb{N}$ have the same growth rate, written $f \equiv g$, iff there exists an integer $t \geq 1$ such that

- $f(n) \leq g(tn)$ *for all* $n \geq 1$, *and*
- $g(n) \leq f(tn)$ *for all* $n \geq 1$.

The quasi-isometry problem

The Main Conjecture

- *The quasi-isometry problem for f.g. groups is universal* $\mathbf{K}_\sigma$.
- *In particular, the quasi-isometry problem is strictly harder than the isomorphism problem.*

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Conjecture

- *The quasi-isometry problem for f.g. groups is strictly harder than the virtual isomorphism problem.*
- *In particular, the virtual isomorphism problem is not universal* $\mathbf{K}_\sigma$.

The virtual isomorphism problem

Theorem (Hjorth-S.T.)

The virtual isomorphism problem for f.g. groups is not universal $\mathbf{K}_\sigma$.

The virtual isomorphism problem

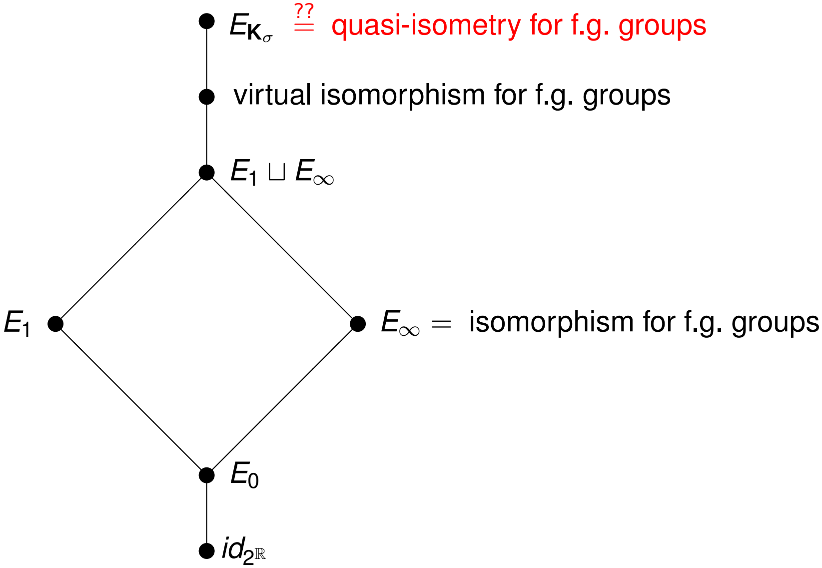
Theorem (Hjorth-S.T.)

The virtual isomorphism problem for f.g. groups is not universal $\mathbf{K}_\sigma$.

Corollary (Hjorth-S.T.)

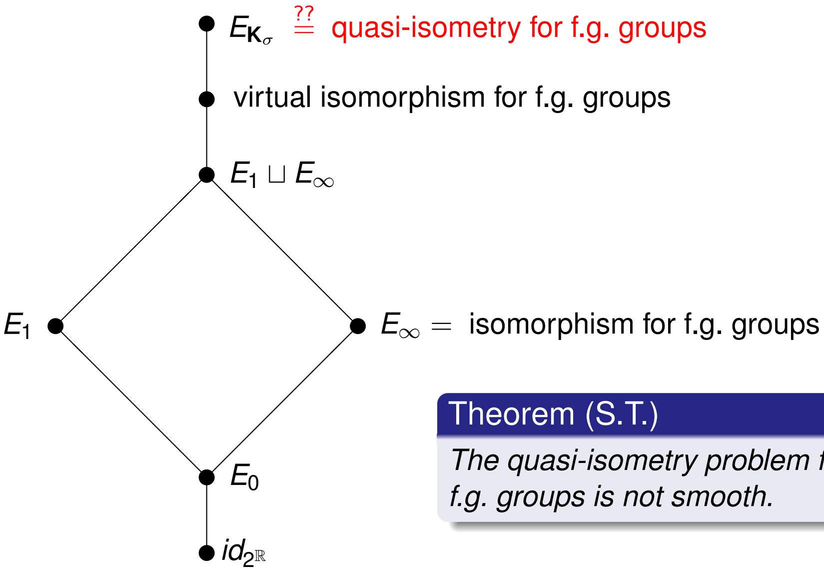
The virtual isomorphism problem for f.g. groups is strictly easier than the quasi-isometry relation for connected 4-regular graphs.

Conclusion



$E_{\mathbf{K}_\sigma} \overset{??}{=}$ quasi-isometry for f.g. groups
virtual isomorphism for f.g. groups
$E_1 \sqcup E_\infty$
E_1
$E_\infty =$ isomorphism for f.g. groups
E_0
$id_{2^{\mathbb{R}}}$

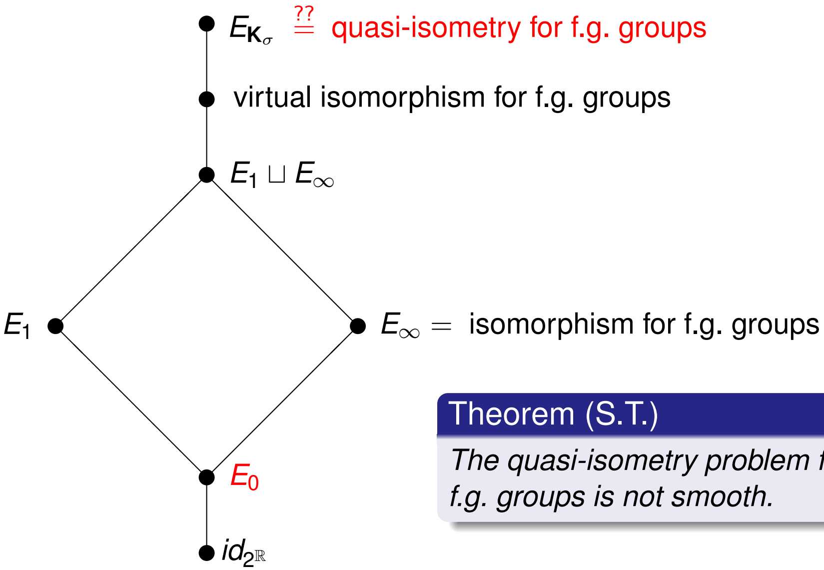
Conclusion



Theorem (S.T.)

The quasi-isometry problem for f.g. groups is not smooth.

Conclusion



Theorem (S.T.)

The quasi-isometry problem for f.g. groups is not smooth.