Crested products

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From Higman-Sims to Urysohn:
a random walk through groups, graphs, designs, and spaces
August 2007

A story of collaboration

Time-line

- Pre-Cambrian:
  - association schemes;
  - transitive permutation groups;
  - direct products (crossing);
  - wreath products (nesting);
  - partitions;
  - orthogonal block structures.

Association schemes

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(iii) if (α, β) is yellow then there are exactly $p^{\text{yellow}}_{\text{red,blue}}$ points γ
such that (α, γ) is red and (γ, β) is blue
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The set of pairs given colour i is called the i-th associate class.

Adjacency matrices

The adjacency matrix A_i for colour i is the $\Omega \times \Omega$ matrix with

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Colour 0 is the diagonal, so

(i) $A_0 = I$ (identity matrix);

(ii) every A_i is symmetric;

(iii) $A_iA_j = \sum_k p_{ij}^k A_k$;

(iv) $\sum_i A_i = J$ (all-1s matrix).

Permutation groups

If G is a transitive permutation group on Ω, it induces a permutation group on $\Omega \times \Omega$.
Give (α, β) the same colour as (γ, δ) iff there is some g in G with $(\alpha^g, \beta^g) = (\gamma, \delta)$.
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association scheme	permutation group
(i) $A_0 = I$	$\Longleftrightarrow$ transitivity
(ii) every A_i is symmetric	$\Longleftrightarrow$ the orbitals are self-paired
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Some of the theory extends if (ii) is weakened to
'if A_i is an adjacency matrix then so is $A_i^\top$',
which is true for permutation groups.

The Bose–Mesner algebra and the character table

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The character table gives each A_i as a linear combination of $S_0, \ldots, S_r$.
Its inverse expresses each S_j as a linear combination of $A_0, \ldots, A_r$.

Direct product (crossing)

association scheme	set	adjacency matrices	index set	Bose–Mesner algebra
$\mathcal{Q}_1$	Ω_1	A_i	$i \in \mathcal{K}_1$	$\mathcal{A}_1$
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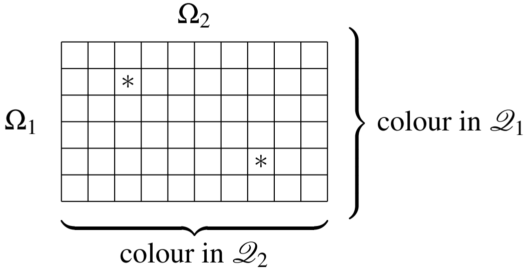
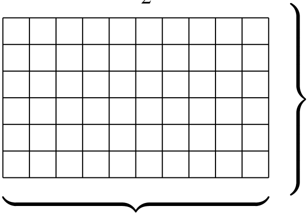


The underlying set of $\mathcal{Q}_1 \times \mathcal{Q}_2$ is $\Omega_1 \times \Omega_2$. The adjacency matrices of $\mathcal{Q}_1 \times \mathcal{Q}_2$ are $A_i \otimes B_j$ for i in $\mathcal{K}_1$ and j in $\mathcal{K}_2$.

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$$

Direct product of permutation groups



If G_1 is transitive on Ω_1 with self-paired orbitals and association scheme $\mathscr{Q}_1$, and
G_2 is transitive on Ω_2 with self-paired orbitals and association scheme $\mathscr{Q}_2$, then
$G_1 \times G_2$ is transitive on $\Omega_1 \times \Omega_2$ with self-paired orbitals and association scheme $\mathscr{Q}_1 \times \mathscr{Q}_2$.

Wreath product (nesting)

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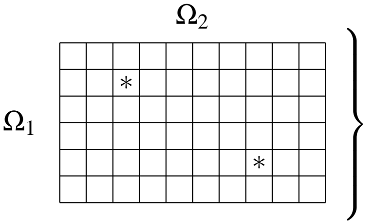

colour in $\mathscr{Q}_1$
if colour $\neq 0$

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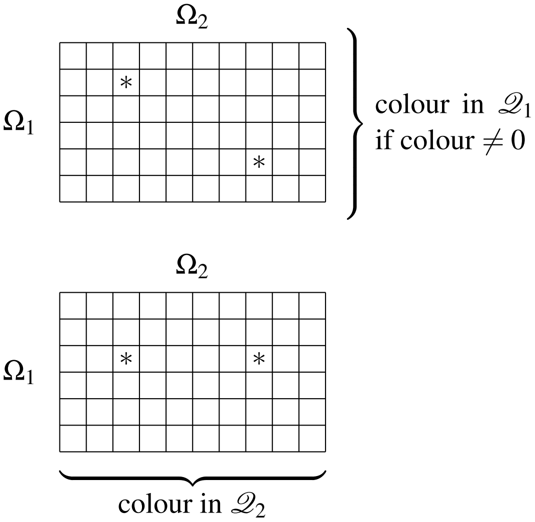

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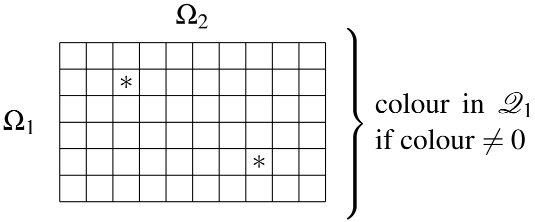

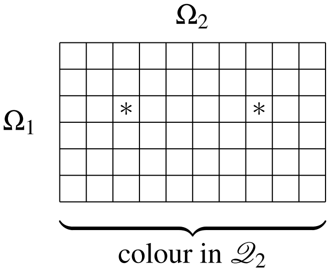

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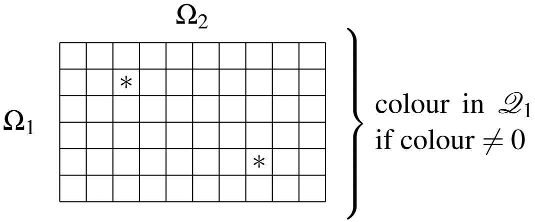

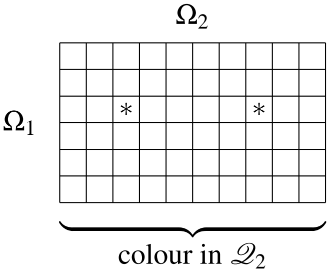

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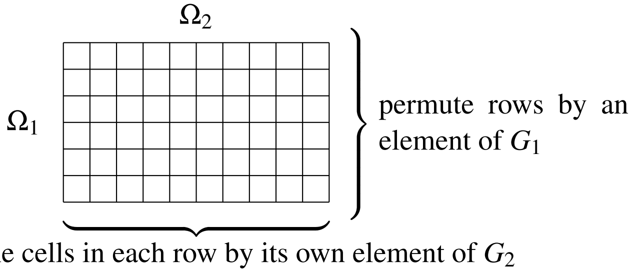
NB $\mathscr{A}_1 \langle I \rangle = \mathscr{A}_1$ and

$$\langle J \rangle \mathscr{A}_2 = \langle J \rangle$$

Wreath product of permutation groups



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Inherent partitions

A partition F of Ω is inherent in the association scheme $\mathcal{Q}$ on Ω if there is a subset $\mathcal{L}$ of the colours such that

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The relation matrix R_F for partition F is the $\Omega \times \Omega$ matrix with

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$$\text{If } F \text{ is inherent then } R_F = \sum_{i \in \mathscr{L}} A_i.$$

Trivial partitions

There are two trivial partitions.

- U is the universal partition, with a single part:

$$R_U = J = \sum_{\text{all } i} A_i.$$

- E is the equality partition, whose parts are singletons.

$$R_E = I = A_0.$$

These are inherent in every association scheme.

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$$\mathscr{A}_1|_F < \mathscr{A}_1 \qquad \text{and} \qquad \langle J \rangle \lhd \mathscr{A}_2$$

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An important paper cited by Bannai and Ito

Theorem (P. J. Cameron, J.-M. Goethals & J. J. Seidel, 1978)

If F is an inherent partition in an association scheme $\mathscr{Q}$ on Ω with Bose–Mesner algebra $\mathscr{A}$ then

1. *the restriction of $\mathscr{Q}$ to any part of F is a subscheme of $\mathscr{Q}$, whose Bose–Mesner algebra is isomorphic to*

$$\operatorname{span}\{A_i : i \in \mathscr{L}\} = \mathscr{A}|_F \ ;$$

2. *there is a quotient scheme on Ω/F, whose Bose–Mesner algebra, pulled back to Ω, is the ideal*

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'The Krein condition, spherical designs, Norton algebras and permutation groups'

"Good stuff in an old paper with one of my five Belgian co-authors and one of my eight Dutch co-authors"

Crested product

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$A_i \otimes B_j$	for i in $\mathcal{L}$ and j in $\mathcal{K}_2$,
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If $F_1 = U_1$ or $F_2 = E_2$, the product is $\mathcal{Q}_1 \times \mathcal{Q}_2$.
If $F_1 = E_1$ and $F_2 = U_2$, the product is $\mathcal{Q}_1 / \mathcal{Q}_2$.

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- July 2001: 18th British Combinatorial Conference, Sussex RAB finds very natural expression for orthogonal block structures

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If R_F commutes with R_H, and F and H are both uniform,
then $R_{F\vee H}$ is a scalar multiple of $R_F R_H$.

Orthogonal block structures

An orthogonal block structure on a finite set Ω is a family $\mathscr{H}$ of uniform partitions of Ω such that

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Theorem

For $i = 1, 2$, let $\mathscr{H}_i$ be an orthogonal block structure on Ω_i with corresponding association scheme $\mathscr{Q}_i$, and let $F_i \in \mathscr{H}_i$. Then

$$\{H_1 \times H_2 : H_1 \in \mathscr{H}_1,\ H_2 \in \mathscr{H}_2,\ H_1 \preceq F_1 \text{ or } F_2 \preceq H_2\}$$

is an orthogonal block structure on $\Omega_1 \times \Omega_2$ and its corresponding association scheme is the crested product of $\mathscr{Q}_1$ and $\mathscr{Q}_2$ with respect to F_1 and F_2.

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- Pre-Cambrian: association schemes; transitive permutation groups; direct products (crossing); wreath products (nesting); partitions; orthogonal block structures
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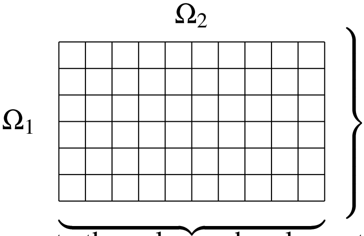
Crested product of permutation groups

F_1 is a partition of Ω_1 preserved by G_1;
F_2 is the orbit partition (of Ω_2) of a normal subgroup N of G_2.



permute rows by an element of G_1

either permute the columns by element of G_2, *or*
for each part of F_1, permute the cells in each row by an element of N

Theorem

If $\mathscr{Q}_i$ is the assocation scheme defined by G_i on Ω_i, for $i = 1, 2$, then the crested product of $\mathscr{Q}_1$ and $\mathscr{Q}_2$ with respect to F_1 and F_2 is the association scheme of the crested product of G_1 and G_2 with respect to F_1 and N.

“I typed my part in LaTeX on my Psion without making a single typo!”

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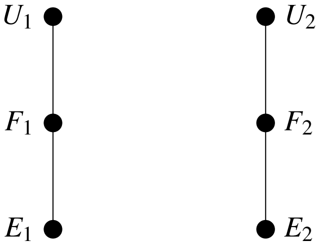

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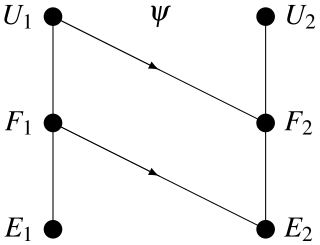

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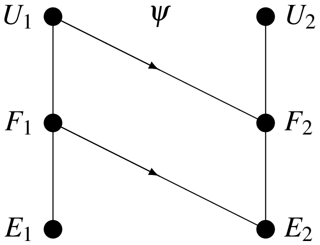

How to do the permutation group theory to match?

"You've gone too far this time. It simply isn't possible to define a way of combining two permutation groups to match what happens in an arbitrary pair of association schemes."

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  RAB does extended crested products of association schemes
- November 2003: PJC and RAB do extended crested products of permutation groups

Extended crested products of permutation groups

A wonderful piece of theory, and the association scheme of the extended crested product of two permutation groups is indeed the extended crested product of the association schemes of the two permutation groups, but this slide is too small to …

The story goes on . . .