Optimal partitions of finite sets:

a report on unfinished work in progress

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- Introduction: set partitions and Bell numbers
- Optimal partitions and the main problem
- Inequalities
- Some near-theorems and computations
- Conclusion

Introduction: the Bell numbers

Throughout n is a natural number, X a finite set of size n.

Recall the Bell number $B_n :=$ number of set-partitions of X.

Recall that

$$\mathrm{e}^{\mathrm{e}^{z}-1} = \sum_{n \geqslant 0} \frac{B_n}{n!}\, z^n.$$

Introduction: a useful function

Define $\lambda(n)$ by $\lambda\,\mathrm{e}^{\lambda} = n$.

Then

$$\lambda(n) = \log n - \log\log n + \frac{\log\log n}{\log n} + O\left(\left(\frac{\log\log n}{\log n}\right)^2\right) .$$

In fact

$$\lambda(n) = \log n - \log\log n + \sum_{k\geqslant 1} \frac{p_k(\log\log n)}{(\log n)^k} ,$$

where $p_k(t)$ is a polynomial of degree k; leading term t^k/k; alternating signs; obtainable by "boot-strapping".

Introduction: estimates for Bell numbers

Many asymptotic estimates for the Bell numbers are known. E.g.

Lovász: $B_n \sim n^{-\frac{1}{2}} \lambda(n)^{n+\frac{1}{2}} \mathrm{e}^{\lambda(n)-n-1}$;

De Bruijn:
$$\frac{\log B_n}{n} = \log n - \log\log n - 1 + \frac{\log\log n}{\log n} + \frac{1}{\log n} + \frac{1}{2}\Big(\frac{\log\log n}{\log n}\Big)^2 + O\left(\Big(\frac{\log\log n}{(\log n)^2}\Big)\right).$$

Optimal partitions: notation

For a partition $\mu \vdash n$ write $\mu = [m_1, m_2, \ldots, m_k]$ to mean:

- μ has m_r parts of size r, so $n = \sum r\, m_r$;
- k is its largest part, so $m_k \geqslant 1$.

Then define

$$A(\mu) := \prod r!^{m_r}\, m_r! \,.$$

So if ρ is a set-partition of X of **shape** μ then $A(\mu) = |\mathrm{Aut}(X;\rho)|$.

Note. $$B_n = \sum_{\mu \vdash n} \frac{n!}{A(\mu)} \,.$$

Optimal partitions: the main problem

Call μ **optimal** if it minimises $A(\mu)$.

Problem. Which partitions μ of n are optimal?

What do they look like asymptotically?

How can we compute them for sizable values of n?

Optimal partitions: basic inequalities

Proposition. If $\mu = [m_1, m_2, \ldots, m_k] \vdash n$ and μ is optimal then

$$\begin{aligned}
\binom{r+s}{r} m_{r+s} &\leqslant (m_r+1)(m_s+1) \quad \text{if } r \neq s\,, \\
\binom{2r}{r} m_{2r} &\leqslant (m_r+1)(m_r+2)\,, \\
\binom{r+s}{r} (m_{r+s}+1) &\geqslant m_r\, m_s \quad \text{if } r \neq s\,, \\
\binom{2r}{r} m_{2r} &\geqslant m_r\,(m_r-1)\,.
\end{aligned}$$

Note. Call these 3-part inequalities: there are also 4-part inequalities, 5-part inequalities, etc.; sometimes useful.

Some near-theorems, I

Near-Theorem. Suppose that n is large, $\mu = [m_1, m_2, \ldots, m_k] \vdash n$, and μ is optimal. Let $c := m_1$. Then

$$c\,\mathrm{e}^{c-2} \leqslant n \leqslant (c+1)\,\mathrm{e}^{c+2}\,,$$
$$c\,\mathrm{e} \leqslant k \leqslant (c+\log c)\,\mathrm{e}\,.$$

Comment. Too crude!

Some near-theorems, II

Near-Theorem [K. Körner]. Suppose that n is large, $\mu = [m_1, m_2, \ldots, m_k] \vdash n$, and μ is optimal. Let $c := m_1$. Then

$$m_1 < m_2 < \cdots < m_{c-1} \leqslant m_c$$

and

$$m_c \geqslant m_{c+1} > m_{c+2} > \cdots > m_{k-1} > m_k > 0\,.$$

That is, μ is **unimodal.**

A method of computation

To seek optimal μ for n in the range $n_0 \leqslant n \leqslant n_1$ do:
find possibilities for m_1;
then find possibilities for m_2;
etc.;
for each n, for those μ that emerge, find smallest $A(\mu)$.

Example [ΠMN, hand calculation] for $\mathbf{n = 10,000}$.
If $m_1 \leqslant 6$ then $n \leqslant 9327$; if $m_1 \geqslant 8$ then $n \geqslant 19,354$; so $\mathbf{m_1 = 7}$.
Then find $\mathbf{m_2 = 25}$ or $\mathbf{m_2 = 26}$;
If $m_2 = 25$ then $\mathbf{m_3 \in \{63, 64\}}$, if $m_2 = 26$ then $\mathbf{m_3 \in \{62, 63, 64\}}$;
etc.

Some computations

Computations [K. Körner, using MAPLE on a PC]:

- All optimal partitions tabulated for $n \leqslant 1100$.
- All optimal partitions tabulated for $10,000 \leqslant n \leqslant 10,100$;
- Method should do $10^5 \leqslant n \leqslant 10^5 + 100$ or even $10^6 \leqslant n \leqslant 10^6 + 100$ in a few hours of computation.

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Conjecture. For very large n, if $\mu = [m_1, m_2, \ldots, m_k] \vdash n$ and μ is optimal then m_r is close to $\lambda(n)^r/r!$.

What does "close to" mean? Certainly $c_1 \leqslant m_r \div \lambda(n)^r/r! \leqslant c_2$, perhaps even $|m_r - \lambda(n)^r/r!| \leqslant c_3$.

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Apply to estimates for Bell numbers B_n .

Happy sixties, Peter