From Higman-Sims to Urysohn: a random walk through groups, graphs, designs, and spaces

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Ambleside
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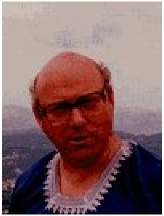  

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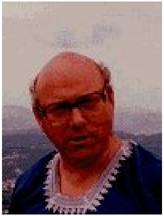  

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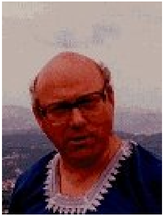  

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The group $\mathrm{PSL}(2, 19)$ acts as a primitive permutation group on 57 points.

The stabiliser of a point is isomorphic to $\mathrm{PSL}(2, 5)$. It has orbits of sizes $1, 6, 20, 30$, and is 2-transitive on the orbit of size 6.

Orbital graphs

We construct a graph of valency 6 on 57 vertices by joining each point α to the points in the G_α-orbit of size 6.

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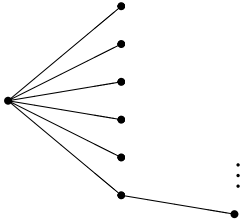

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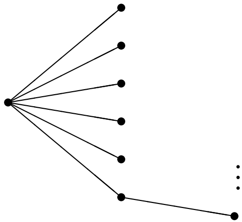

The automorphism group of the graph is transitive on paths of length 2. So there are no triangles, and the ends of the paths of length 2 starting at α form a single G_α-orbit of size $6 \cdot 5/k$ for some k.

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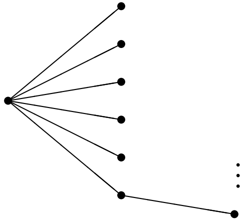

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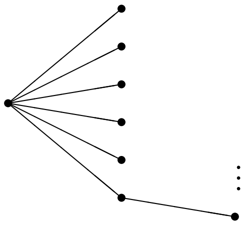

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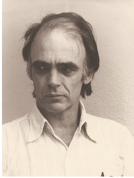
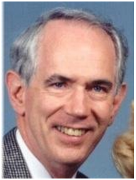
Triangle-free graphs with a lot of symmetry will appear very often in this talk!

The Higman–Sims group

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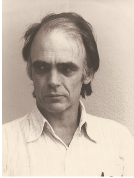
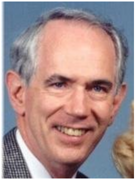


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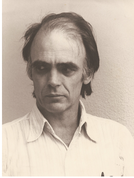 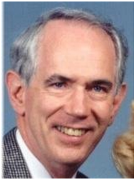 

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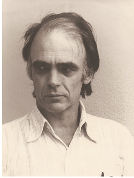
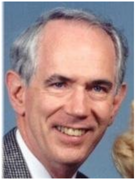


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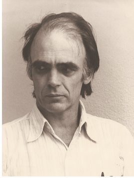 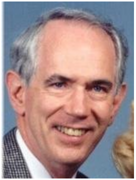 

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The Higman–Sims group acts transitively on 3-claws, on 4-cycles, and on paths of length 3 not contained in 4-cycles.
(The graph was constructed earlier by Dale Mesner, who never thought to look at its automorphism group. The group was constructed in a different action by Graham Higman.)

Designs

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In other words, it is a 3-$(22, 6, 1)$ design, the famous Witt design. (This is how Higman and Sims constructed the graph!)

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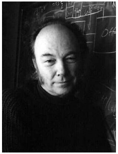
Note that, if β is a point of the design, then the number of points different from β and the number of blocks incident with β are both 21. In other words, D is an extension of a symmetric design (the projective plane of order 4).

Cameron's Theorem

Theorem

If a 3-(v, k, λ) design is an extension of a symmetric 2-design then one of the following holds:

- $v = 4(\lambda + 1), k = 2(\lambda + 1)$ (*Hadamard design*);
- $v = (\lambda + 1)(\lambda^2 + 5\lambda + 5), k = (\lambda + 1)(\lambda + 2)$;
- $v = 112, k = 12, \lambda = 1$ (*extension of projective plane of order* 10);
- $v = 496, k = 40, \lambda = 3$.

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 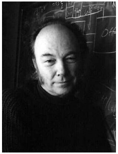 

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 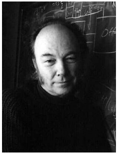 

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The only new thing we know now is that there is no projective plane of order 10 (Lam *et al.*).

Fun with permutation groups

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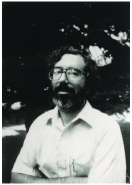
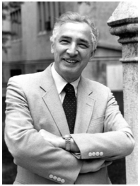
The proof makes a long detour. First, a counterexample preserves a parallelism of the t-subsets of $\{1, \ldots, n\}$. From this one constructs a symmetric triangle-free graph which is locally like a cube. Then one shows that it is a quotient of a cube by a subspace of $\mathrm{GF}(2)^n$. This subspace turns out to be an extension of a perfect $(t-1)$-error-correcting code; the theorem of van Lint and Tietäväinen identifies the code and hence the group.

The Cameron–Kantor Theorem





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 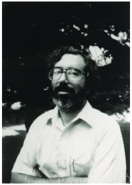 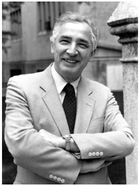 

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 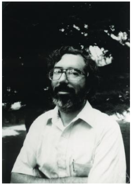 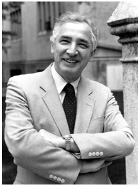 

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 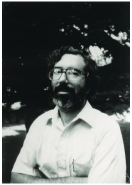 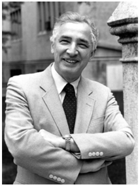 

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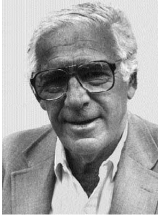
The proof used a lot of nice geometry, including spreads in projective space and generalised polygons (for which the Feit–Higman theorem applies).
But this kind of fun was soon to come to an end!

CFSG



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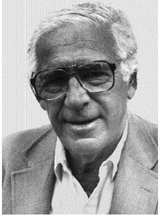

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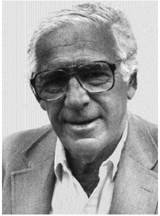

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In particular, all 2-transitive groups were now "known" modulo CFSG, so proving theorems like those on the last two slides would no longer bring promotion and pay!

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Theorem

Let G be an infinite permutation group which is t-set transitive for all natural numbers t. Then either

- *G is t-transitive for all natural numbers t; or*
- *there is a linear or circular order preserved or reversed by G.*

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Woodrow showed that, with some trivial exceptions, the first and third properties characterise H.

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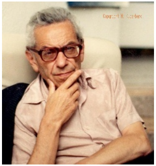

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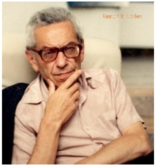  

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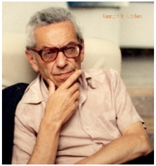  

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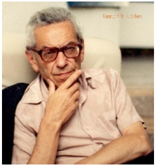  

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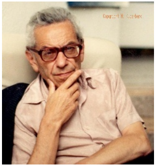

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A graph with a cyclic automorphism is a Cayley graph for $\mathbb{Z}$, say $\mathrm{Cay}(\mathbb{Z}, S \cup (-S))$ for some set S of positive integers; in other words, the vertex set is $\mathbb{Z}$, and we join x and y if and only if $|x - y| \in S$. The cyclic shift $x \mapsto x + 1$ is an automorphism.

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In other words, R is the random Cayley graph for $\mathbb{Z}$.

Cayley graphs and B-groups

More generally, Ken Johnson and I showed:

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Let X be a countable group with the property that X cannot be written as the union of finitely many translates of square root sets and a finite set. Then, with probability 1, a random Cayley graph for X is isomorphic to R.

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Corollary

A countable group satisfying the conditions of the theorem above is not a B-group.

Cyclic automorphisms of H

let S be a set of positive integers. Then $\mathrm{Cay}(\mathbb{Z}, S \cup (-S))$ is triangle-free if and only if S is sum-free, that is, $x, y \in S \Rightarrow x + y \notin S$.

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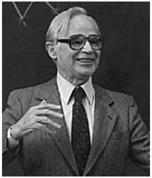
So H has many cyclic automorphisms.

Combinatorial number theory



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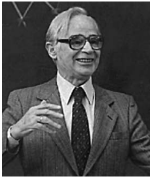  

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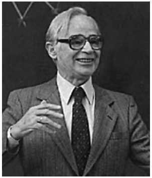

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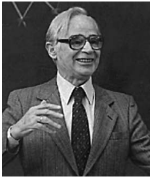  

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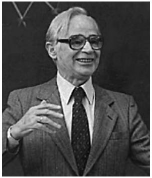  

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There is no density version of Schur's theorem. The odd numbers have density $\frac{1}{2}$ and clearly form a sum-free set.

But what if . . . ?

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What happens if we use measure instead of category?

Random sum-free sets

Choose S by considering the natural numbers in turn. When considering n, if $n = x + y$ with $x, y \in S$, then $n \notin S$; otherwise toss a fair coin to decide.

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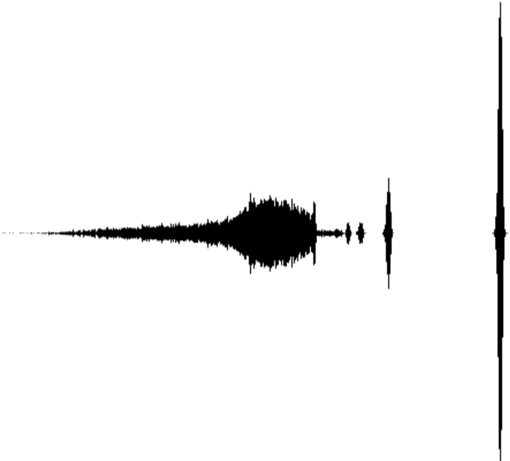
Experimentally, the density of a large random sum-free set looks like this:



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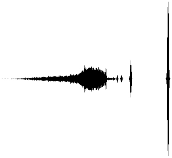

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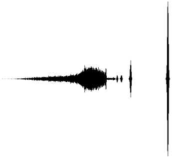

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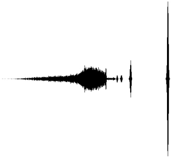

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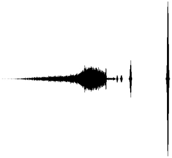

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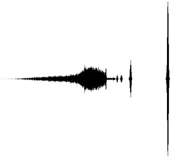

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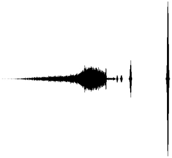

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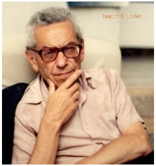
Maybe the density spectrum has a continuous part???

Erdős number 1



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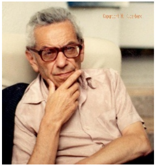

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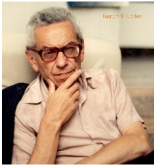 

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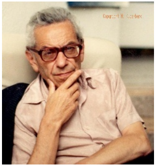

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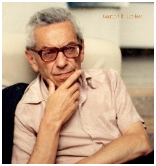

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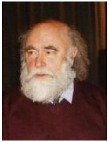
The numbers $c_e \approx 6.0$ and $c_o \approx 6.8$ are two of "Cameron's sum-free set constants" in Steven Finch's book *Mathematical Constants*.

The Urysohn space



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 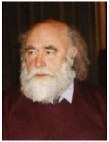

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 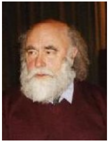

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A Polish space is a complete separable metric space. In a posthumous paper in 1927, Urysohn proved:

Theorem

There is a Polish space $\mathbb{U}$ with the properties

- $\mathbb{U}$ *is universal (it contains an isometric copy of every Polish space);*
- $\mathbb{U}$ *is homogeneous (any isometry between finite subsets of* $\mathbb{U}$ *can be extended to an isometry of the whole space).*

Moreover, a space with these properties is unique up to isometry.

Metric spaces

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In the first two cases we can modify the construction to produce the analogue of Henson's graph (i.e. no equilateral triangles with side 1), or a bipartite graph (all triangles have even perimeter).

Problem

What are the countable homogeneous metric spaces?

The Urysohn space

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Vershik showed that "almost all" Polish spaces are isomorphic to $\mathbb{U}$, in each of two senses. A Polish space is the completion of a countable metric space, and the latter can be constructed by adding points one at a time, so the notions of Baire category and measure can both be applied to the product space. Now $\mathbb{U}$ is residual in the sense of Baire category, and is the random Polish space for any of a wide variety of measures on the set of possible points that can be added at each stage.

Isometries of $\mathbb{U}$

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All we know is that the elementary abelian 2-group has this property but the elementary abelian 3-group does not.

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The closure of the countable elementary abelian 2-group with dense orbits is an elementary abelian 2-group acting transitively on U.