Intersecting Permutations and Set Partitions

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What is the structure of such a family of maximum size?

- $2^{[n]}$ set of all subsets of $[n] = \{1, \ldots, n\}$.

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$\mathcal{A} \subseteq 2^{[n]}$ is intersecting if $A \cap B \neq \emptyset$ for any distinct $A, B \in \mathcal{A}$.

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- $k > \frac{n}{2}$. Then every two k-subset intersect. **Answer.** $\binom{n}{k}$.
- $k \leq \frac{n}{2}$. Take $\mathcal{A} = \{A : |A| = k, x \in A\}$ some fixed x. Then $|\mathcal{A}| = \binom{n-1}{k-1}$. Can we do better?

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We are interested in the following function:

$M(n,k,t) = \max\{|\mathcal{A}| : \mathcal{A} \in I(n,k,t)\}$, for $2k - t < n$.

Construction - Frankl Families

For $0 \leq r \leq \frac{n-t}{2}$, let

$$\mathcal{F}_r = \mathcal{F}_{n,k,t,r} = \{F \in \binom{[n]}{k} : |F \cap [1, t+2r]| \geq t + r\}.$$

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- $|\mathcal{F}_0| = \binom{n-t}{k-t}$;
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Theorem (Erdős-Ko-Rado, 1961)

For $1 \leq t \leq k$ *and* $n \geq n_0(k, t)$,

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$-$ for $n > (t+1)(k-t+1)$, optimum families $= \mathcal{F}_0$ (up to permutations);

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Technique - Shifting Operation

The well-known (i,j)-shift S_{ij} is defined as follows:

$$\triangleleft_{ij}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\} & \text{if } \ i \notin A, j \in A \\ A & \text{otherwise.} \end{cases}$$

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Shifting preserves the problem!

$\mathcal{A}$ $\rightarrow$ repeated shifting operations $\cdots \rightarrow$ $\mathcal{A}^*$

$\uparrow$

has 'nice' properties

Sometimes, we can even 'undo' the shiftings to say something about $\mathcal{A}$!

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- e.g. $d_H(51234, 12534) = 3$. Formally,

$$\begin{aligned} d_H(g,h) &= |\{i : g(i) \neq h(i)\}| \\ &= n - |\{i : g(i) = h(i)\}| \\ &= n - (g^{-1}h), \end{aligned}$$

where $(g) = |\{i : g(i) = i\}|$.

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Problems.

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1. What is the size of a largest t-intersecting family of permutations?

2. Can we characterize such extremal families?

The Case $t = 1$

Theorem (Deza-Frankl, 1977)

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Note: The bound is sharp. Take a point stabilizer.

Theorem (Cameron-Ku, 2003; Larose-Malvenuto, 2004)

Equality holds if and only if $\mathcal{A}$ *is a coset of a point stabilizer, i.e.*

$$\mathcal{A} = \{g : g(x) = y\},$$

for some $x, y \in [n]$.

Design Theoretic Approach

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Main problem: such design might not exist.

Graphical Interpretation

Theorem (Clique-Coclique Bound)

$G = (V, E)$ vertex-transitive. C complete subgraph, I independent set. Then

$$|C| \cdot |I| \quad \leq \quad |V(G)|. \tag{1}$$

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The statement of equality can be used to characterize extremal families.

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Observe that

- G is vertex-transitive.
- An independent set $=$ an 1-intersecting family.
- A complete subgraph $=$ rows of a Latin rectangle.

A complete subgraph of size n exists - just take the rows of a Latin square.

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By Clique-Coclique Bound,

$$n \cdot |\mathcal{A}| \leq n!,$$

where $\mathcal{A}$ is an 1-intersecting family (independent set of G) of permutations of $[n]$.

Deza-Frankl (1977): Let q be a prime power.

'Design'	n	t	upper bound for t-intersecting family of permutations	
Latin square	all	1	$(n-1)!$	(sharp)
$AGL(1,q)$	q	2	$(n-2)!$	(sharp)
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Rick Wilson (personal communication): $n = 10$, $t = 2$. Largest 2-intersecting family has size $(n-2)!$.

Conjecture. (Deza-Frankl, 1977)

$\mathcal{A} \subseteq \mathrm{Sym}([n])$ t-intersecting. Then, for $n \geq n_0(t)$,

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Example. Take $n = 8$, $t = 4$, and let $\mathcal{A}$ consists of the identity and all transpositions intechanging i and j, $i \neq j$. Easy to check that $|\mathcal{A}| = 1 + \binom{8}{2} > (8-4)!$ and $\mathcal{A}$ is 4-intersecting.

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For $n \geq n_0(t) = O(t^2)$, $|\mathcal{A}| \leq (n-t)!$,

with equality if and only if $\mathcal{A}$ is a coset of the stabilizer of t points.

What happens if we replace $\mathrm{Sym}([n])$ by the alternating group $\mathrm{Alt}(n)$ or direct product of symmetric groups $\mathrm{Sym}([n_1]) \times \cdots \mathrm{Sym}([n_q])$?

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Theorem (Ku-Wong, 2007)

Let $n \geq 2$. $\mathcal{A} \subseteq \mathrm{Alt}(n)$ 1-intersecting. Then

$$|\mathcal{A}| \leq (n-1)!/2.$$

If $n \neq 4$, then equality holds iff $\mathcal{A} = \{g \in \mathrm{Alt}(n) : g(x) = y\}$,

some $x, y \in [n]$.

The condition $n \neq 4$ is necessary:

$$\{(1,2,3,4),(1,3,4,2),(2,3,1,4)\}.$$

Theorem (Ku-Wong, 2007)

Let $2 \leq m \leq n$*.* $\mathcal{A} \subseteq \mathrm{Sym}(\Omega_1) \times \mathrm{Sym}(\Omega_2)$ *1-intersecting,* $|\Omega_1| = m$*,* $|\Omega_2| = n$*. Then*

$$|\mathcal{A}| \leq (m-1)!n!.$$

Moreover,

Theorem (Ku-Wong, 2007)

Let $2 \le m \le n$. $\mathcal{A} \subseteq \mathrm{Sym}(\Omega_1) \times \mathrm{Sym}(\Omega_2)$ *1-intersecting,* $|\Omega_1| = m$, $|\Omega_2| = n$. *Then*

$$|\mathcal{A}| \le (m-1)!n!.$$

Moreover,

- *if* $m < n$, $(m, n) \neq (2, 3)$, *then equality holds iff* $\mathcal{A} = \{(g, h) : g(x) = y\}$, $x, y \in \Omega_1$;

- *if* $m = n$, $(m, n) \neq (3, 3)$, *then equality holds iff* $\mathcal{A} = \{(g, h) : g(x) = y\}$, $x, y \in \Omega_1$ *or* $\mathcal{A} = \{(g, h) : h(x) = y\}$, $x, y \in \Omega_2$.

Theorem (Ku-Wong, 2007)

Let $2 \leq n_1 = \cdots = n_p < n_{p+1} \leq \cdots \leq n_q$.
$\mathcal{A} \subseteq \mathrm{Sym}(\Omega_1) \times \cdots \times \mathrm{Sym}(\Omega_q)$ *1-intersecting,* $|\Omega_i| = n_i$. *Then*

$$|\mathcal{A}| \leq (n_1 - 1)! \prod_{i=2}^{q} n_i!.$$

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Except for the following cases:

- $n_1 = \cdots = n_p = 2 < n_{p+1} = 3 \leq n_{p+2} \leq \cdots \leq n_q$, $1 \leq p < q$;
- $n_1 = n_2 = 3 \leq n_3 \leq \cdots \leq n_q$;
- $n_1 = n_2 = n_3 = 2 \leq n_4 \leq \cdots \leq n_q$;

Equality holds iff $\mathcal{A} = \{(g_1, \ldots, g_q) : g_i(x) = y\}$, *some* $1 \leq i \leq p$, $x, y \in \Omega_i$.

Cycle-Intersecting Permutations

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The stabilizer of t points is t-cycle-interseting.

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We are interested in the following function:

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Fixing Operation - Shifting Analogue

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- If $\mathcal{A} \subseteq \mathrm{Sym}(n)$ then $\mathrm{F}_{ij}(\mathcal{A}) \subseteq \mathrm{Sym}(n)$;
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$U(n, k) =$ set of all k-partitions of $[n]$ such that k divides n. (Uniform Set Partitions).

Examples.

- $n = 6,\ k = 3 : 12 - 345 - 6 \in P(6,3)$.
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Problem 1. What is the maximum size of a t-intersecting family in $B(n)$ or $P(n, k)$ or $U(n, k)$?

Problem 2. Can we characterize the extremal families for these problems?

k-partitions

A construction of t-intersecting families in $P(n,k)$:

$$\mathcal{P} = \{P \in P(n,k) : \{1\}, \{2\}, \dots, \{t\} \in P\}.$$

$$|\mathcal{P}| = S(n-t, k-t),$$

where $S(n,k)$ is the Stirling Number of the Second Kind.

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Theorem (P. L. Erdős-Székely, 2000)

Suppose $\mathcal{A} \subseteq P(n, k)$ is t-intersecting. For $n \geq n_0(t)$,

$$|\mathcal{A}| \leq |\mathcal{P}| = S(n - t, k - t).$$

Uniform Set Partitions

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Let $n \geq k \geq 1$, $n = kc$. Then

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Equality holds iff $\mathcal{A}$ is isomorphic to

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Theorem (Meagher-Moura, 2005)

Let $n \geq k \geq t \geq 1$, $n = kc$. $\mathcal{A} \subseteq U(n,k)$ t-intersecting. Then, for $n \geq n_0(k,t)$ or $n \geq n_0(c,t)$ (when $c \geq t+2$),

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Conjecture. (Peter Cameron) For sufficiently large n,

$$M(n, t) \leq B(n - t).$$

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There are 16 set partitions of [6] with 4 or more singletons.

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Non-Trivial 1-Cycle-Intersecting Families

Hilton-Milner type construction.

$$\mathcal{H}^{\dagger} = \{g \in \mathrm{Sym}(n) : g(1) = 1, g(i) = i \text{ for some } i > 2\}$$

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$$|\mathcal{H}^{\dagger}| = (n-1)! - d(n-1) - d(n-2) + 1 \sim (1 - \tfrac{1}{e})(n-1)!$$

$d(n)$ is the number of derangements on n elements.

Theorem (Ku-Renshaw, 2007)

Let $n \geq 194$ and $\mathcal{A} \subseteq \mathrm{Sym}(n)$ be non-trivial 1-cycle-intersecting. Then

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Conjecture.

- Equality holds iff $\mathcal{A} = \mathcal{H}^{\dagger}$ (up to permutations).

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$\mathcal{A}$ non-trivial cycle-intersecting

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Case I. $\mathrm{F}_{ab}(\mathcal{R})$ is contained in the trivial family for some $a \neq b$.

Case II. Otherwise.

Case I. **Case I never occurs.**

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apply ij – fixing $i, j \notin \{a, b\}$
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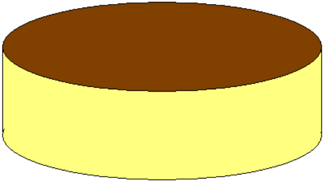
Finally, use LYM-type inequalities and the structure of $\mathcal{F}$ to show that

$$|\mathcal{G}| \leq \sum_{F \in \mathcal{F}} (n - |F|)! \leq |\mathcal{H}^{\dagger}|. \qquad \blacksquare$$

Happy 60th Birthday, Peter!



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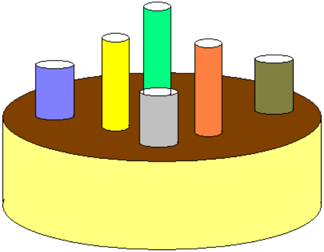

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