Happy Birthday Peter!

Constructing Point Imprimitive Designs

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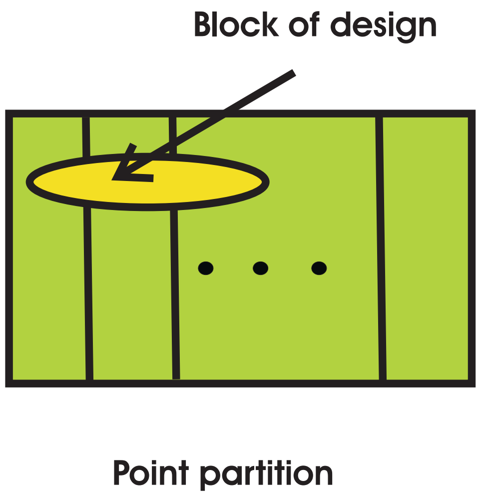
Theme of lecture: Designs based on a specified point partition



Constructions: that allow symmetry to be determined

Usually lots of symmetry: automorphism group transitive on blocks and points, and preserving the specified point partition.

Context: finite t-designs

$t-(v,k,\lambda)$-design $\mathcal{D}=(\Omega,\mathcal{B})$ ($1 \le t \le k \le v$, $\lambda \ge 1$): consists of

a 'point set' Ω of v points;
a 'block set' $\mathcal{B}$ of k-element subsets of Ω (called blocks)
each t-element subset of Ω lies in λ blocks

$\mathcal{D}$ non-trivial: $t < k < v-t$; usually $t \ge 2$

Automorphism of $\mathcal{D}$: a permutation of Ω leaving $\mathcal{B}$ invariant

Automorphism group: $G \le \mathrm{Aut}\,(\mathcal{D}) \le \mathrm{Sym}\,(\Omega)$, G transitive on $\mathcal{B}$

Set-up and some restrictions

$t-(v,k,\lambda)$ design $\mathcal{D}=(\Omega,\mathcal{B})$ with $G\leq \mathrm{Aut}\,(\mathcal{D})$
G block-transitive (and hence point-transitive)
G-invar't partition $\mathcal{C}$ of Ω with $d:=|\mathcal{C}|>1$, and $c:=|\Delta|>1$ $(\Delta\in\mathcal{C})$

Delandtsheer-Doyen bound 1989: $\exists$ positive integers x, y such that

$$v=cd \;=\; \frac{\binom{k}{2}-x}{y}\cdot\frac{\binom{k}{2}-y}{x}\leq\left(\binom{k}{2}-1\right)^2$$

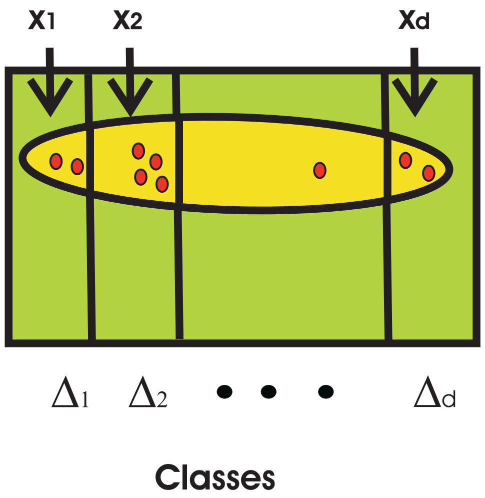
Consequence: given k, only finitely many block-transitive, point-imprimitive t-designs with block size k

Cameron/CEP 1993: $t\leq 3$

Cameron-Praeger construction 1993

To determine the possibilities for: $t,\ k,\ c,\ d.$
Suppose $\mathcal{D} = (\Omega, \mathcal{B})$ t-design: G-block trans, G-invar $\mathcal{C}$ with

$$\mathcal{C} = \{\Delta_1, \ldots, \Delta_d\},\ c = |\Delta_i| > 1.$$



Consider the multiset
$\mathrm{x} := \{x_i, \ldots, x_d\}$
Since G is block transitive
x **is same for all blocks.**

Key observation: if we

Replace G by: $\widehat{G} = \mathsf{Sym}\,(\Delta) \wr S_d =$ stabiliser of $\mathcal{C}$ in $\mathsf{Sym}\,(\Omega)$.
Replace $\mathcal{B}$ by: $\widehat{\mathcal{B}} = \widehat{G}$-images of blocks of $\mathcal{B}$.
Then: new design $\widehat{\mathcal{D}} = (\Omega, \widehat{\mathcal{B}})$ is also t-design, $\mathcal{C}$ is $\widehat{G}$-invariant, and $\widehat{\mathcal{D}}$ is $\widehat{G}$-block-transitive. **Same t, k, c, d.**

$\widehat{\mathcal{D}}$ can be defined for any $c, d, \mathbf{x}$: block size $k := \sum_{i=1}^{d} x_i$
Always: block-transitive, point-imprimitive 1-design

2-design iff: $\sum_{i=1}^{d} \binom{x_i}{2} = \binom{k}{2} \frac{(c-1)}{cd-1}$

3-design iff: 2-design and $\sum_{i=1}^{d} \binom{x_i}{3} = \binom{k}{3} \frac{(c-1)(c-2)}{(cd-1)(cd-2)}$

Some comments

In these examples: number of blocks usually much larger then v; λ usually large

Somewhat similar story for symmetric designs: that is, $|\mathcal{B}| = v$ for example, $\mathcal{D}$ symmetric and $\lambda = 1$ $\Leftrightarrow$ $\mathcal{D}$ a projective plane (note in general $|\mathcal{B}| \geq v$)

Strengthen transitivity assumption: to flag-transitive, that is, transitivity on incident point-block pairs.

Bounds for flag-transitive point-imprimitive designs

Davies 1987: $\mathcal{D}$ is a flag-transitive, point-imprimitive $2-(v,k,\lambda)$ design $\Rightarrow$ k (and hence v) bounded by some function of λ (no explicit function given)

O'Reilly Regueiro 2005: $\mathcal{D}$ is a flag-transitive, point-imprimitive **symmetric** $2-(v,k,\lambda)$ design $\Rightarrow$ either $k \leq \lambda(\lambda-2)$ or

$$(v,k,\lambda) = (\lambda^2(\lambda+2), \lambda(\lambda+1), \lambda)$$

Regueiro 2005: $\lambda \leq 4$ $\Rightarrow$ 4 feasible (v,k,λ)

Regueiro: Examples for small λ

(15,8,4)	≥ 1 example	Regueiro 2005
(16,6,2)	2 examples	Hussain 1945
(45,12,3)	no information	refered to Mathon& Spence ($>$ 3700 examples, no symmetry info'n)
(96,20,4)	≥ 1 example	Regueiro 2005

Further analysis gave more information:

(15,8,4)	unique example	PG(3,2) (CEP+Zhou)
(45,12,3)	**unique example**	**CEP**
(96,20,4)	≥ 4 examples	Law, Reichard et al

The $2-(45,12,3)$ example

Point partition: 5 classes Δ_i of size 9

(full) Automorphism Group: $G = Z_3^4 \cdot [10.4] \leq \mathrm{A\Gamma L}(1,81)$

Structure induced on class Δ: Affine plane $\mathrm{AG}(2,3)$

Line of $\mathcal{D}$: union of a line from four of the $\mathrm{AG}(2,3)$

Peter Cameron: recognised $\mathcal{D}$ as possibly an example of combinatorial construction method given by Sharad Sane 1982

Sane 1982: no information about automorphisms

Cameron & CEP: new construction
$\mathcal{D} = (\Omega, \mathcal{B})$ with point partition $\mathcal{C}$

Features of construction. **Ingredients:**

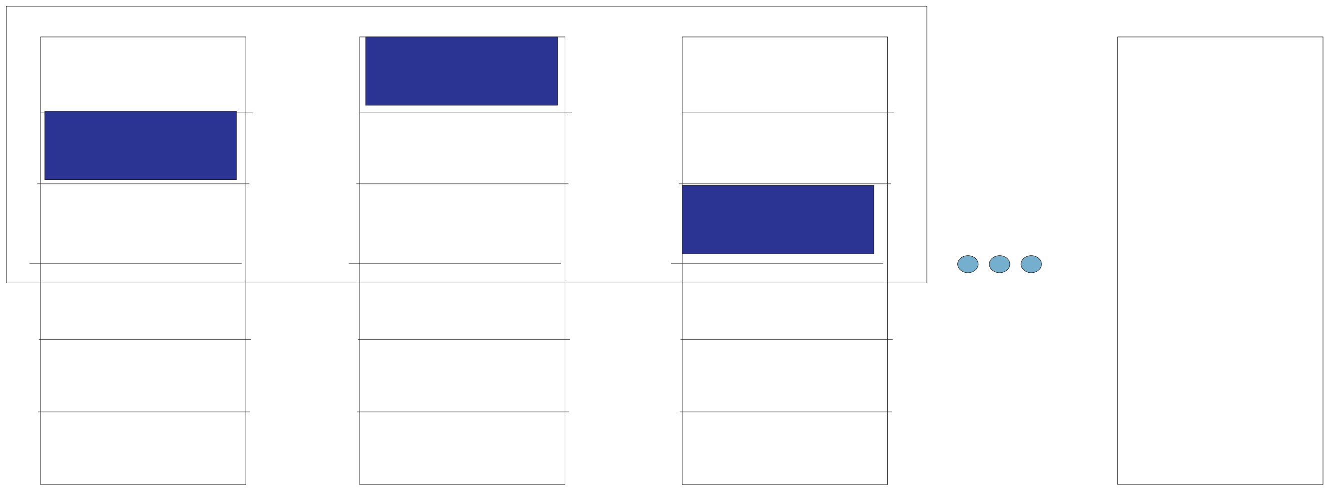
0: 2-design $\mathcal{D}_0 = (\Delta_0, \mathcal{L}_0)$ with block set partitioned into a set $\mathcal{P}_0$ of parallel classes. **Induced on each class of $\mathcal{C}$**

1: symmetric 2−design $\mathcal{D}_1 = (\mathcal{C}, \mathcal{L}_1)$, and for each blocks $\beta \in \mathcal{L}_1$, a bijection $\psi_\beta : \mathcal{P}_0 \to \beta$. **Induced on the set $\mathcal{C}$**

2: transversal design $\mathcal{D}_2 = (\cup_{P \in \mathcal{P}_0} P, \mathcal{L}_2)$. **used to select blocks from parallel classes of $\mathcal{P}_0$ to form blocks of $\mathcal{D}$**

Blocks of the 'big design' $\mathcal{D} = (\Omega, \mathcal{B})$

$\mathcal{D}$: point set $\Omega = \mathcal{C} \times \Delta_0$ block set $\mathcal{B} \leftrightarrow \mathcal{L}_1 \times \mathcal{L}_2$



So have block partition, one block class for each $\beta \in \mathcal{L}_1$

Parameters

Design $\mathcal{D}$: is a 1-design, not necessarily a 2-design

Parameters of $\mathcal{D}$: given in terms of parameters of $\mathcal{D}_0, \mathcal{D}_1, \mathcal{D}2$

Conditions on $\mathcal{D}$: to be a 2-design; to be symmetric
(in terms of parameters)

Automorphisms

Our interest - determine: $G := \mathrm{Aut}\,(\mathcal{D}) \cap (\mathrm{Sym}\,(\Delta_0) \wr \mathrm{Sym}\,(\mathcal{C}))$
that is, $G = \mathrm{Aut}\,(\mathcal{D}) \cap$ (Stabiliser of point partition)

What we found:

1. $G \leq G^* := \mathrm{Aut}^*(\mathcal{D}_0) \wr \mathrm{Aut}\,(\mathcal{D}_1)$ where $\mathrm{Aut}^*(\mathcal{D}_0)$ is subgroup of $\mathrm{Aut}\,(\mathcal{D}_0)$ preserving $\mathcal{P}_0$
2. G equals subgroup of G^* satisfying specific property concerning maps ψ_β and $\mathcal{D}_2$
3. G acts faithfully on blocks as subgroup of $\mathrm{Aut}\,(\mathcal{D}_2) \wr \mathrm{Sym}\,(\mathcal{L}_1)$

When can $\mathcal{D}$ be G-flag-transitive?

Recall: on points $G \leq \mathrm{Aut}^*(\mathcal{D}_0) \wr \mathrm{Aut}(\mathcal{D}_1)$

and on blocks $G \leq \mathrm{Aut}(\mathcal{D}_2) \wr \mathrm{Sym}(\mathcal{L}_1)$

Necessary and sufficient conditions:

1. G flag-transitive on $\mathcal{D}_1$
2. stabiliser of each point class flag-transitive on $\mathcal{D}_0$
3. stabiliser of each block class flag-transitive on $\mathcal{D}_2$
4. additional property on stabiliser of flag of $\mathcal{D}_1$ and block of $\mathcal{D}_2$ (transitive on corresponding block of $\mathcal{D}_0$)

Additional Examples I

New $2-(1408, 336, 80)$ **design:** admitting flag-transitive, point-imprimitive action of $[4^6]\, 3.M_{22} < \mathrm{AGL}(6,4)$

$1408 = 4^3 \cdot 22 \quad \text{and} \quad 336 = 21 \cdot 16$

$\mathcal{D}_0 = \mathrm{AG}_2(3,4)$ (points and planes)

$\mathcal{D}_1 =$ degenerate design on 22 points

$\mathcal{D}_2$ has 21 groups size 4; block size 21

Design constructed and group properties checked using GAP

Additional Examples II

Symplectic design $S^-(n)$: admits a flag-transitive, point-imprimitive action of Z_2^{2n}.GL$(n, 2)$.

Point set $V = V(2n, 2)$
Block set $2^{n-1}(2^n - 1)$ quadratic forms (of - type) polarising to given symplectic form
Full automorphism group Aut$(S^-(n)) = Z_2^{2n}$.Sp$(2n, 2)$.

We give alternate construction exhibiting imprimitivity system preserved by Z_2^{2n}.GL$(n, 2)$ - decomposition of V as sum of two maximal totally singular subspaces. [For $S^+(n)$ get point-primitive Z_2^{2n}.GU$(n, 2)$].

Summary

Additional structure imposed on block-transitive designs by an imprimitivity system on points.

Theoretical bounds on block size, or parameter λ.

Also interesting examples and constructions.