

VERTEX-TRANSITIVE STRONGLY REGULAR GRAPHS IN THE SWITCHING CLASS OF DOUBLY TRANSITIVE TWO-GRAPHS

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ABSTRACT. Let G be a permutation group that acts 2-transitively on the finite set V and let $\mathcal{T} = (V, T)$ be a two-graph whose automorphism group contains G . In this paper, we classify those strongly regular graphs Γ with vertex set V whose automorphism group is a transitive maximal subgroup of G and whose associated two-graph is $\mathcal{T}$. In doing so, we obtain a new family of vertex-transitive strongly regular graphs whose associated two-graph arises from $P\Sigma L(2, q)$.

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1. INTRODUCTION

A *strongly regular graph* with parameters (v, k, λ, μ) is a k -regular graph with v vertices, where any two adjacent vertices have λ common neighbours, and any two non-adjacent vertices have μ common neighbours. A strongly regular graph is said to be *primitive* if both it and its complement are connected, and is *imprimitive* otherwise; it is well-known that the only imprimitive strongly regular graphs are the complete multipartite graphs $K_{t,t,\dots,t}$ or their complements. We refer to the monograph of Brouwer and Van Maldeghem [4] for more details.

A *two-graph* is a pair (V, T) , where T is a set of unordered triples of a vertex set V , such that every (unordered) quadruple from V contains an even number of triples in T ; see [3, Chapter 10] for more information. Given a graph $\Gamma = (V, E)$, the set of triples T of the vertex set V whose induced subgraph has an odd number of edges is the *associated two-graph* of Γ . A two-graph is *trivial* if either T is empty or contains all unordered triples of V . Two-graphs were introduced by G. Higman and studied in more detail by Taylor [21, 22].

Two simple graphs are said to be *switching equivalent* if we can obtain one from the other by the *switching* of edges and non-edges between subsets of vertices. We will be primarily interested in *Seidel switching*, introduced by Seidel [19], which is a method to modify a given graph to obtain another which is cospectral to the original (i.e. their adjacency matrices have the same eigenvalues). This operation is of interest as two simple graphs are switching equivalent if and only if they have the same associated two-graph.

Finite nontrivial two-graphs with 2-transitive automorphism groups G have been classified by Taylor [23]:

- 1) **Affine polar type:** $G \cong \mathbb{F}_2^{2m} \rtimes Sp(2m, 2)$, $m \geq 2$;
- 2) **Symplectic type:** $G \cong Sp(2m, 2)$, $m \geq 3$;
- 3) **Linear type:** $G \cong P\Omega L(2, q)$, for $q \equiv 1 \pmod{4}$;
- 4) **Unitary type:** $G \cong PU(3, q)$, for $q \geq 5$ odd;
- 5) **Ree type:** $G \cong Ree(q) \rtimes \text{Aut}(\mathbb{F}_q)$, for $q = 3^{2e+1}$, $e \geq 1$;
- 6) **Sporadic type:** $G \cong HS$ or $G \cong Co_3$.

In particular, G is either almost simple or a primitive group of affine type. We note that some restrictions are imposed to ensure that there are no overlaps between the six classes. We exclude $Sp(4, 2)$ since $Sp(4, 2) \cong P\Omega L(2, 9) \cong S_6$, which acts 2-transitively on 10 points; this action yields the two-graph of the Petersen graph (in our language, this is a two-graph of linear type). We also exclude the unitary group $PU(3, 3)$ and the smallest Ree group $Ree(3)$: both are maximal subgroups of $Sp(6, 2)$, and the corresponding two-graphs are, in fact, a symplectic type two-graph on 28 points.

Let $G^{(\infty)}$ denote the unique largest perfect subgroup of G . Equivalently, $G^{(\infty)}$ is the smallest normal subgroup of G with a solvable factor group. The main result of this paper is the following.

Theorem 1.1. *Let $\Gamma = (V, E)$ be a primitive strongly regular graph with associated two-graph $\mathcal{T} = (V, T)$. Write $H = \text{Aut}(\Gamma)$ and $G = \text{Aut}(\mathcal{T})$. Assume that G is 2-transitive, and that H is a transitive maximal subgroup of G that does not contain $G^{(\infty)}$. Then G , H , and Γ are as in Table 1.*

In all but one case of Theorem 1.1, the subgroup H is maximal in G . When $G = PU(3, 5) = PSU(3, 5) \rtimes S_3$, then $H \cong S_7$ is a maximal subgroup of $PSU(3, 5) \rtimes C_2$, hence a maximal subgroup not containing $G^{(\infty)} = PSU(3, 5)$. All the graphs in Table 1 are known, with the exception of the infinite family associated with $P\Omega L(2, q)$; to the authors' knowledge these graphs are new, and the existence of these graphs is established in Section 6. $ASp(2m, 2)$ denotes the affine symplectic group, i.e. the semidirect product

TABLE 1. Vertex-transitive strongly regular graphs in 2-transitive two-graphs

G	H	Γ	Properties	References
$ASp(2m, 2)$	$\mathbb{F}_2^{2m} \rtimes O^\pm(2m, 2)$	$VO_{2m}^\pm(2)$	for $m \geq 2$; affine polar graph	[4, Sec. 3.3]
$Sp(2m, 2)$	$O^\pm(2m, 2)$	$NO_{2m}^\pm(2)$	for $m \geq 3$; nonsingular points	[4, Sec. 3.1.2]
	$Sp(m, 4) \rtimes C_2$	$NO_{m+1}^\pm(4)$	for $m \geq 4$ even; two classes of graphs	[4, Sec. 3.1.4]
$P\Sigma L(2, q)$	$C_{\frac{q+1}{2}} \rtimes C_{4e}$	new srg & its complement	for $q = p^{2e} > 9$, $p \equiv 3 \pmod{4}$; the complement $\bar{\Gamma}$ is in this class as well	Section 6
$Sp(6, 2)$	$G_2(2)$	$U_3(3)$ -graph	$ V = 36$; subconstituent of the Hall–Janko graph	
$Sp(8, 2)$	S_{10}	generalized Johnson graph $J(10, 3, 1)$	$ V = 120$	
	$PSL(2, 17)$	$srg(136, 63, 30, 28)$	$ V = 136$	[1, 8]
$P\Sigma L(2, 9)$	S_5	Petersen graph & its complement	$P\Sigma L(2, 9)$ has two conjugacy classes of S_5	
$P\Gamma U(3, 5)$	S_7	Goethals graph, $srg(126, 50, 13, 24)$	parameter uniqueness; H is maximal in G not containing $G^{(\infty)} = PSU(3, 5)$	[14]
HS	M_{22}	$srg(176, 70, 18, 34)$	parameter uniqueness	[7], [9], [12, Thm. 5.2]

$G \cong \mathbb{F}_2^{2m} \rtimes Sp(2m, 2)$. The term *parameter uniqueness* used in Table 1 means that the parameters determine the strongly regular graph up to isomorphism.

The paper is organized as follows. In Section 2, we will give all the necessary preliminaries on (finite) 2-transitive groups and two-graphs. Section 3 reviews the known infinite families of strongly regular graphs which appear in the paper, along with the associated groups, as well as the sporadic and exceptional examples. In Section 4, we explain the algebraic language of cocycles and coboundaries to represent graphs and two-graphs, while in Section 5, we define the linear and affine actions of the automorphism groups on the switching class. Section 6 is devoted to the new infinite family of graphs with linear type two-graphs. Finally, Sections 7 and 8 contain proofs of the nonexistence of certain graphs, using deep results on the classification of maximal subgroups of almost simple groups.

2. PRELIMINARIES

2.1. Group actions and 2-transitivity. A permutation group G on a set A is a subgroup of the symmetric group $Sym(A)$. An action of G on A may be regarded as a homomorphism from G into the symmetric group. The image of an element $a \in A$ under the permutation corresponding to $g \in G$ will be written a^g . We say that G is *transitive* in its action on A if for all $a_1, a_2 \in A$, there exists $g \in G$ with $a_1^g = a_2$. The relation $\sim$ on A where $a_1 \sim a_2$ if and only if $a_1^g = a_2$ for some $g \in G$ is an equivalence relation, whose equivalence classes are called *orbits*. Therefore, G is transitive on each of its orbits. For a positive integer t , a permutation group G is *t-transitive* on A if its induced action on the set of ordered t -tuples of distinct elements of A is transitive.

The *orbitals* of G are the orbits of its action on $A \times A$, and the number of orbitals is called the *rank* of G . For $|A| \geq 1$, G has rank at least 2, with equality if and only if G is 2-transitive. Moreover, G is 2-transitive if and only if the stabilizer of each element $a \in A$ is transitive on $A \setminus \{a\}$. We will not need the full classification of finite 2-transitive groups: Table 2 lists the 2-transitive actions that occur in the present paper; for the full classification, we refer to Cameron [5]. For finite simple groups, we follow the notation used in the ATLAS [24] (see also Wilson [25]). Extensions of groups will be written as follows: $H \times K$ will denote a *direct product*; $H \rtimes K$ will denote a *semidirect product* with normal subgroup H and complement K .

TABLE 2. The 2-transitive groups occurring in the paper.

Type	G	Degree	Point stabilizer G_v
Affine polar	$ASp(2m, 2)$	2^{2m}	$Sp(2m, 2)$
Symplectic	$Sp(2m, 2)$	$2^{2m-1} \pm 2^{m-1}$	$O^\pm(2m, 2)$
Linear	$P\Sigma L(2, q)$	$q + 1$	P_1
Unitary	$PTU(3, q)$	$q^3 + 1$	P_1
Ree	$Ree(q) \rtimes Aut(\mathbb{F}_q)$	$q^3 + 1$	P
Sporadic	HS	176	$PSU(3, 5) \rtimes C_2$
Sporadic	Co_3	276	$McL \rtimes C_2$

2.2. Two-graphs.

Definition 2.1 (Two-graph). A *two-graph* is a pair (V, T) , where T is a set of unordered triples of a vertex set V , such that every (unordered) quadruple from V contains an even number of triples from T . The two-graph is called *regular* if each pair of vertices is in a constant number of triples.

Definition 2.2 (Associated two-graph). Given a graph $\Gamma = (V, E)$, the set of triples T of the vertex set V , whose induced subgraph has an odd number of edges, forms a two-graph on the set V . The two-graph $\Omega(\Gamma) = (V, T)$ is called the *associated two-graph* of Γ .

Let $\mathcal{T} = (V, T)$ be a two-graph. The *descendant* of $\mathcal{T}$ with respect to the vertex $v \in V$ is the graph $\mathcal{T}_v = (V, E)$ where E consists of pairs $\{w, z\}$ such that $\{v, w, z\} \in T$. By definition, v is an isolated vertex of the descendant. It is not hard to see that $\Omega(\mathcal{T}_v) = \mathcal{T}$, which shows that every two-graph can be represented as the associated two-graph of a graph.

Definition 2.3 (Seidel switching [19]). Given a graph $\Gamma = (V, E)$ and a subset Y of the vertex set V , the operation of *switching* Γ with respect to Y consists of replacing all edges from Y to $V \setminus Y$ by nonedges, and all such nonedges by edges, while leaving the edges within Y or outside Y unchanged.

The subset Y is called a *switching set*. We say that a switching set Y is *regular of degree d* if the subgraph induced by Y is d -regular. Switching defines an equivalence relation on the set of all graphs with a given n -element vertex set, each equivalence class containing 2^{n-1} graphs (since switching with respect to a set and its complement yield the same graph, while switching with respect to two sets is the same as switching with respect to their symmetric difference). The equivalence classes are called *switching classes*. It is well known that a pair of graphs are switching equivalent if and only if they have the same associated two-graph. Consequently, it is meaningful to speak of the switching class of a two-graph.

We now give a few important results regarding the parameters of two-graphs and related strongly regular graphs.

Result 2.4 ([4, Proposition 1.1.2]). *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) and spectrum $k^1 r^f s^g$. Let Δ be a strongly regular graph of valency $\ell > k$ switching equivalent to Γ' . Then:*

- (1) Δ has spectrum $\ell^1 r^{f-1} s^{g+1}$,
- (2) $\frac{v}{2} = k - s = \ell - r$,
- (3) $k - r = 2\mu$,
- (4) $\frac{v}{2} = 2k - \lambda - \mu$.
- (5) Any switching set from Γ to Δ has size $\frac{1}{2}v$ and is regular of degree $k - \mu$.

Result 2.5 ([3, Proposition 10.3.3]). *Let Γ be a strongly regular graph with parameters (v, k, λ, μ) associated with a regular two-graph. Then either:*

- (1) the graph Γ is switched into a strongly regular graph with the same parameters if and only if every vertex outside the switching set S is adjacent to half of the vertices of S ,
- (2) the graph Γ is switched into a strongly regular graph with parameters $(v, k + c, \lambda + c, \mu + c)$ where $c = \frac{v}{2} - 2\mu$ if and only if the switching set S has size $\frac{v}{2}$ and is regular of degree $k - \mu$.

Result 2.6 ([3, Theorems 10.3.1 and 10.3.2]). *Let $\mathcal{T}$ be a regular two-graph on v vertices and regularity a .*

- (i) For any descendant $\mathcal{T}_v$, $\mathcal{T}_v - v$ is a $\text{srg}(v - 1, a, \frac{3a-v}{2}, \frac{a}{2})$.
- (ii) Let Γ be a $\text{srg}(v, k, \lambda, \mu)$ in the switching class of $\mathcal{T}$. Then k is a root of the polynomial

$$X^2 - \left(\frac{v}{2} + a\right)X + \frac{a}{2}(v - 1),$$

and

$$\lambda = k + \frac{a - v}{2}, \quad \mu = k - \frac{a}{2}.$$

2.3. 2-transitive two-graphs. In this section, we review Taylor's classification of finite two-graphs with 2-transitive automorphism groups.

Trivially, the only graphs with 2-transitive automorphism groups are complete graphs or their complements, since edges cannot be mapped to non-edges and vice-versa. However, regular two-graphs may have 2-transitive automorphism groups; see [22, Theorem 6.1]. Let G be a (finite) group acting 2-transitively on a set V of vertices and consider a (regular) two-graph $\mathcal{T}$ on V such that $G \leq \text{Aut}(\mathcal{T})$. By [23], we know a complete classification of such two-graphs, by their automorphism groups.

Result 2.7 ([23, Theorem 1]). *If $(V, \mathcal{T})$ is a regular two-graph admitting an automorphism group G , acting 2-transitively on V , then one of the following occurs:*

- (1) G has a normal subgroup N , and letting $n = |V|$:
 - $N \cong \text{PSL}(2, q)$, $q \equiv 1 \pmod{4}$, $n = q + 1$;
 - $N \cong \text{PSU}(3, q)$, q odd, $n = q^3 + 1$;
 - $N \cong \text{Ree}(q)$, $q = 3^{2e+1}$, $n = q^3 + 1$;
 - $N \cong \text{Sp}(2m, 2)$, $m \geq 3$, $n = 2^{2m-1} \pm 2^{m-1}$;
 - $N \cong HS$, $n = 176$;
 - $N \cong Co_3$, $n = 276$.
- (2) G is of affine type.

Result 2.8 ([23, Theorem 2]). *For each group in Result 2.7, there is (up to isomorphism) a unique pair of complementary two-graphs admitting G as a 2-transitive group of automorphisms. Using the notation above, such a group is:*

- (1) **Linear type:** $G \cong \text{P}\Sigma\text{L}(2, q)$, for $q \equiv 1 \pmod{4}$;
Unitary type: $G \cong \text{P}\Gamma\text{U}(3, q)$, for $q \geq 5$ odd;
Ree type: $G \cong \text{Ree}(q) \rtimes \text{Aut}(\mathbb{F}_q)$, for $q = 3^{2e+1}$, $e \geq 1$;

TABLE 3. The parameters v , a and a' of 2-transitive two-graphs

Type	G	v	a	a'
Affine	$ASp(2m, 2)$	2^{2m}	$2^{2m-1} - 2$	2^{2m-1}
Symplectic	$Sp(2m, 2)$	$2^{2m-1} \pm 2^{m-1}$	$2^{2m-2} \pm 2^{m-1} - 2$	2^{2m-2}
Linear	$P\Sigma L(2, q)$	$q + 1$	$(q - 1)/2$	$(q - 1)/2$
Unitary	$PTU(3, q)$	$q^3 + 1$	$(q - 1)(q^2 + 1)/2$	$(q + 1)(q^2 - 1)/2$
Ree	$Ree(q)$	$q^3 + 1$	$(q - 1)(q^2 + 1)/2$	$(q + 1)(q^2 - 1)/2$
Sporadic	HS	176	72	102
	Co_3	276	162	112

Symplectic type: $G \cong Sp(2m, 2)$, $m \geq 3$;

Sporadic type: $G \cong HS$ or $G \cong Co_3$.

- (2) **Affine polar type:** The automorphism group is the semidirect product $\mathbb{F}_2^{2m} \rtimes Sp(2m, 2)$, where $\mathbb{F}_2^{2m}$ is the additive group of the vector space of dimension $2m$ over $\mathbb{F}_2$.

The symplectic groups $Sp(2, 2) \cong S_3$ and $Sp(4, 2) \cong S_6$ act 3-transitively on 3 and 6 points; hence, the associated two-graphs are trivial. The 2-transitive actions of degree 10 of $Sp(4, 2)$ and $P\Sigma L(2, 9)$ are equivalent, and the corresponding two-graphs are the same. In their 2-transitive actions of degree 28, $PTU(3, 3)$ and $Ree(3) \cong P\Gamma L(2, 8)$ are (maximal) subgroups of $P\Sigma Sp(6, 2)$. The two-graphs obtained from these three groups are isomorphic.

The 2-transitive automorphism group G does not define the regular two-graph $\mathcal{T}$ uniquely, only up to taking the complement. If $v = |V|$ is the number of vertices, then one has two choices a, a' for the regularity of $\mathcal{T}$; $a + a' = v - 2$. Typically, one uses the values for a and a' as listed in Table 3.

3. KNOWN GRAPHS AND THEIR TWO-GRAPHS IN TABLE 1

In this section, we present the constructions of the known graphs that occur in Table 1, except for the infinite class of graphs associated with the linear type two-graphs. Here, we have three infinite classes of graphs, and a further five exceptional or sporadic examples. At the end of the section, we identify the associated two-graphs of these graphs (Proposition 3.1).

3.1. Affine polar graphs $VO_{2m}^\pm(2)$. Let $m \geq 2$. Let Q be a nondegenerate quadratic form of type ± 1 in the vector space $V = \mathbb{F}_2^{2m}$, and let $\langle x, y \rangle = Q(x + y) + Q(x) + Q(y)$ be the associated symplectic bilinear form. In $VO_{2m}^\pm(2)$, two different vectors x, y are adjacent when $Q(x + y) = 0$. This yields a strongly regular graph with parameters

$$\begin{aligned} v &= 2^{2m}, & k &= (2^m \mp 1)(2^{m-1} \pm 1), \\ \lambda &= 2(2^{m-1} \mp 1)(2^{m-2} \pm 1), & \mu &= 2^{m-1}(2^{m-1} \pm 1). \end{aligned}$$

The full automorphism group is $Aut(VO_{2m}^\pm(2)) \cong \mathbb{F}_2^{2m} \rtimes O^\pm(2m, 2)$, consisting of the semidirect product of the orthogonal group and the additive translation group, as in the first row of Table 1. See [4, Section 3.3] for more information.

3.2. Graphs $NO_{2m}^\pm(2)$ on nonsingular points over $\mathbb{F}_2$. Let $m \geq 3$. The graph $NO_{2m}^\pm(2)$ is the graph whose vertex set is $PG(2m - 1, 2) \setminus Q^\pm(2m - 1, 2)$, and two vertices are adjacent if and only if the corresponding points lie on a line tangent to the quadric $Q^\pm(2m - 1, 2)$. $NO_{2m}^\pm(2)$ is a strongly regular graph with the following

parameters:

$$\begin{aligned} v &= 2^{2m-1} \mp 2^{m-1}, & k &= 2^{2m-2} - 1, \\ \lambda &= 2^{2m-3} - 2, & \mu &= 2^{2m-3} \pm 2^{m-2}. \end{aligned}$$

The automorphism group of $NO_{2m}^{\pm}(2)$ is the orthogonal group $O^{\pm}(2m, 2)$. See [4, Section 3.1.2] for more information on this graph.

3.3. Graphs $NO_{2m+1}^{\pm}(4)$ on nonsingular points of one type. Let $Q(2m, 4)$ be a nonsingular parabolic quadric in $PG(2m, 4)$. Let $\mathcal{H}^{\pm}$ be the set of nontangent hyperplanes intersecting the quadric in a nonsingular quadric of type ± 1 . The graph $NO_{2m+1}^{\pm}(4)$ is the graph whose vertex set is $\mathcal{H}^{\pm}$, and two vertices H_1 and H_2 are adjacent if and only if $H_1 \cap H_2 \cap Q(2m, 4)$ is degenerate. The graph $NO_{2m+1}^{\pm}(4)$ is strongly regular, and its automorphism group is $Aut(NO_{2m+1}^{\pm}(4)) \cong Sp(2m, 4) \rtimes C_2$. This graph is described in [4, Section 3.1.4].

In [18], the authors proved that the two families $NO_{4m}^{\mp}(2)$ and $NO_{2m+1}^{\pm}(4)$ are switching equivalent. The parameters of $NO_{2m+1}^{\pm}(4)$ are:

$$\begin{aligned} v &= 2^{4m-1} \pm 2^{2m-1}, & k &= (2^{2m-2} \pm 1)(2^{2m} \mp 1), \\ \lambda &= 2^{4m-3} \pm 2^{2m-2} - 2, & \mu &= 2^{4m-3} \pm 2^{2m-1}. \end{aligned}$$

The two-graphs associated with both $NO^{\pm}(4m, 2)$ and $NO_{2m+1}^{\mp}(4)$ are $\mathcal{X}_{4m}^{\pm} = (X, T)$ with

$$\begin{aligned} X &= \{a \in \mathbb{F}_2^{4m} \mid \Theta(a) = 1\} \\ T &= \{\{a, b, c\} \mid a \neq b \neq c \neq a, \langle a, b \rangle + \langle a, c \rangle + \langle b, c \rangle = 0\}. \end{aligned}$$

where Θ is a nondegenerate quadratic form of type $\pm$, and $\langle \cdot, \cdot \rangle$ is its associated symplectic bilinear form.

3.4. Sporadic and exceptional graphs.

3.4.1. The S_5 -invariant graph: The Petersen graph and its complement. The Petersen graph with parameters $(10, 3, 0, 1)$ and its complement with parameters $(10, 6, 3, 1)$ are the only primitive strongly regular graphs on 10 vertices. The Petersen graph is isomorphic to the tangent graph $NO_4^-(2)$, while its complement is isomorphic to both $NO_3^+(4)$ and the triangular graph $T(5)$; see Sections 3.2 and 3.3. The automorphism groups of both graphs are isomorphic to $S_5 \cong O^-(4, 2) \cong Sp(2, 4) \rtimes C_2$. A simple calculation shows that the associated two-graph consists of 60 unordered triples, with automorphism group $P\Sigma L(2, 9) \cong Sp(4, 2) \cong S_6$.

3.4.2. The $G_2(2)$ -invariant graph. The group $G_2(2)$ of Lie type is isomorphic to $PSU(3, 3) \rtimes C_2$. It has a primitive permutation representation on 36 vertices. An $srg(36, 14, 4, 6)$ is known to exist with automorphism group $G_2(2)$ and point stabilizer $PGL(2, 7)$. This graph is also called the $U_3(3)$ -graph; see [4, Section 10.14] for further details. Although the graph is not determined by its parameters alone, it is the unique graph with these parameters that admits a transitive rank-3 action of $G_2(2)$ of degree 36.

3.4.3. Goethals' S_7 -invariant graph. There is a unique strongly regular graph with parameters $(v, k, \lambda, \mu) = (126, 50, 13, 24)$. The full automorphism group is S_7 . Existence is due to Goethals; uniqueness is due to Coolsaet and Degraer (see [4, Section 10.42], [6] and [14]).

3.4.4. *The S_{10} -invariant graph.* When $H = S_{10}$, we obtain the strongly regular generalized Johnson graph $J(10, 3, 1)$. Recall that the generalized Johnson graphs are the graphs $J(n, k, m)$ whose vertices are the k subsets of $\{1, 2, \dots, n\}$, with two vertices A, B joined by an edge if and only if $|A \cap B| = m$. The graphs $J(n, k, 1)$ are typically not strongly regular, but Mathon and Klin showed that $J(10, 3, 1)$ is an exception (see [4, Section 1.3.6]). In particular, $J(10, 3, 1)$ is strongly regular with parameters $(120, 63, 30, 36)$, and is therefore cospectral to $NO_8^+(2)$, but is not isomorphic to it since the automorphism group is S_{10} .

3.4.5. *The $PSL(2, 17)$ -invariant graph.* Let $G = Sp(8, 2)$, in its 2-transitive action of degree 136. Then the group $H = PSL(2, 17)$ is a transitive maximal subgroup. In [1] the first author and his students obtained a strongly regular graph with parameters $(136, 63, 30, 28)$ and full automorphism group $PSL(2, 17)$ (acting primitively with rank 12), as a result of a computer search using the GRAPE package for GAP [10, 20]; the existence of such a graph was also established independently in [8]. The graph is cospectral with but is not isomorphic to $NO_8^-(2)$.

3.4.6. *The M_{22} -invariant graph.* Let $G = HS$ be the Higman–Sims group, acting on 176 vertices. Then the maximal Mathieu subgroup $H = M_{22}$ gives rise to a $srg(176, 70, 18, 34)$. This graph was constructed by Goethals and Seidel [12, Theorem 5.2]; see also [4, Section 10.51] and [7] for further details. The graph is uniquely determined by its parameters [9]. This graph is an induced subgraph of the strongly regular McLaughlin graph $srg(275, 112, 30, 56)$. Indeed, M_{22} is a maximal subgroup of the McLaughlin group McL with orbit lengths 22, 77 and 176.

We note that the Higman–Sims group HS has a maximal subgroup isomorphic to S_8 . Starting from the Higman symmetric design on 176 points (which has HS as its automorphism group) Brouwer [2] constructed a $srg(176, 49, 12, 14)$ with this action of S_8 as its automorphism group (see also [4, Section 10.50]). However, this action is intransitive, so this graph cannot arise in our classification. Furthermore, its associated two-graph is not regular (this can be seen from the eigenvalues of the graph) and thus cannot be the Higman–Sims two-graph.

3.5. Identifying the associated two-graphs.

Proposition 3.1. *The strongly regular graphs described in this section have the following associated two-graphs:*

- (i) *The affine polar graphs $VO_{2m}^\pm(2)$ have the affine two-graph with full automorphism group $ASp(2m, 2)$.*
- (ii) *The graph $NO_{2m}^\pm(2)$ and, when m is even, the graph $NO_{m+1}^\pm(4)$; have the symplectic two-graph $\mathcal{X}_{2m}^\pm$.*
- (iii) *The Petersen graph and its complement have the linear two-graph on 10 vertices, whose full automorphism group is $PTL(2, 9) \cong Sp(4, 2) \cong S_6$.*
- (iv) *The $U_3(3)$ -graph has the symplectic two-graph on 36 vertices, whose full automorphism group is $Sp(6, 2)$.*
- (v) *The Goethals graph has the Hermitian two-graph on 126 vertices, whose full automorphism group is $PTU(3, 5)$.*
- (vi) *The graph $J(10, 3, 1)$ has the symplectic two-graph on 120 vertices, whose full automorphism group is $Sp(8, 2)$.*
- (vii) *The $PSL(2, 17)$ -invariant $srg(136, 63, 30, 28)$ has the symplectic two-graph on 136 vertices, whose full automorphism group is $Sp(8, 2)$.*
- (viii) *The M_{22} -invariant $srg(176, 70, 18, 34)$ has the Higman–Sims two-graph on 176 vertices, whose full automorphism group is HS .*

Proof. Part (i) follows from the standard description of the affine polar two-graph, while part (ii) follows from [18]. Part (iii) is immediate. Parts (iv) and (v) follow from the references given above. Parts (vi), (vii), and (viii) are verified computationally (using GAP and GRAPE [10, 20]) by constructing the associated two-graphs and computing their full automorphism groups. $\square$

We conclude this section by remarking that parts (vi) and (vii) of Proposition 3.1 are new results: the fact that $J(10, 3, 1)$ and the $PSL(2, 17)$ -graph have symplectic two-graphs shows that the former is switching equivalent to $NO_8^+(2)$ and the latter is switching equivalent to $NO_8^-(2)$, which was not previously known.

4. THE COBOUNDARY OPERATOR

Taylor [23] introduced the notion of the coboundary operator to study graphs and their two-graphs in an algebraic language. To provide a uniform algebraic framework for describing switching classes, we will adopt this concept.

In this section, V is a nonempty set, and $\mathbb{F}_2 = \{0, 1\}$ is the finite field of order 2. For a positive integer k , a k -cochain is a totally symmetric mapping $f : V^{k+1} \rightarrow \mathbb{F}_2$ such that $f(x_0, \dots, x_k) = 0$ whenever two of $x_0, \dots, x_k$ are equal. The set $V^{(k,*)}$ of k -cochains is an $\mathbb{F}_2$ -linear space of dimension $\binom{|V|}{k+1}$. Their support

$$\text{supp}(f) = \{\{x_0, \dots, x_k\} \mid f(x_0, \dots, x_k) = 1\}$$

uniquely defines the k -cochains. Following Taylor [22], we define the *coboundary operator*

$$\partial : V^{(k,*)} \rightarrow V^{(k+1,*)}$$

by

$$\partial f(x_0, \dots, x_{k+1}) = \sum_{i=0}^{k+1} f(x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_{k+1}).$$

A k -cochain of the form ∂f is called a k -coboundary, and a k -cochain f such that $\partial f = 0$ is called a k -cocycle. An easy calculation shows that $\partial^2 f = 0$; hence, a k -coboundary is a k -cocycle. The converse is also true, viz., a k -cocycle is a k -coboundary. To see this, choose $v \in V$ and define a “contracting homotopy” $\Delta_v : V^{(k+1,*)} \rightarrow V^{(k,*)}$ by $(\Delta_v f)(x_0, \dots, x_k) = f(x_0, \dots, x_k, v)$, where $f \in V^{(k+1,*)}$. A short calculation shows that for $f \in V^{(k,*)}$ we have

$$f = \Delta_v(\partial f) + \partial(\Delta_v f).$$

In particular, if $\partial f = 0$, then $f = \partial(\Delta_v f)$.

Now observe that a graph with vertex set V can be regarded as a 1-cochain $f : V^2 \rightarrow \mathbb{F}_2$, where $f(x, y) = 1$ if and only if x and y are adjacent. Similarly, if τ is a 2-cochain, then $\mathcal{T} = (V, \text{supp}(\tau))$ is a two-graph. Conversely, if $\mathcal{T}$ is a two-graph, then $f = \Delta_v \tau$ is a graph in its switching class. More precisely, the graph of the 1-cochain f has $\mathcal{T}$ as the associated two-graph if and only if $\partial f = \tau$. In other words, the switching class of τ consists of the 1-cochains f such that $\partial f = \tau$.

Let G be a group acting on V . Then G has an induced action on cochains:

$$f^g(x_0, \dots, x_k) = f(x_0^{g^{-1}}, \dots, x_k^{g^{-1}}).$$

This action is $\mathbb{F}_2$ -linear, and it is equivalent to the natural action of G on graphs and two-graphs with vertex set V . The *automorphism group* of f is the set $\text{Aut}(f)$ of those permutations π of V for which $f = f^\pi$.

Evidently, $\partial(f^g) = (\partial f)^g$, which implies that G acts on cocycles. The following proposition demonstrates the action of G on switching classes.

Proposition 4.1. *Let $\tau \in V^{(k+1,*)}$ be a cocycle and define*

$$\mathcal{W}(\tau) = \{h \in V^{(k,*)} \mid \partial h = \tau\}.$$

Then $\mathcal{W}(0)$ and $\mathcal{W}(0) \cup \mathcal{W}(\tau)$ are $\mathbb{F}_2$ -linear spaces and $\text{Aut}(\tau)$ induces a linear action on them.

Proof. We only have to show that any $g \in \text{Aut}(\tau)$ preserves $\mathcal{W}(\tau)$. This follows from $\partial(h^g) = (\partial h)^g$, $0^g = 0$, and $\tau^g = \tau$. $\square$

We illustrate this method on the class of affine polar graphs $VO_{2m}^\pm(2)$; see [4, Section 3.3] and Section 3.1. $VO_{2m}^\pm(2)$ is a strongly regular graph with parameters

$$\begin{aligned} v &= 2^{2m}, & k &= (2^m \mp 1)(2^{m-1} \pm 1), \\ \lambda &= 2(2^{m-1} \mp 1)(2^{m-2} \pm 1), & \mu &= 2^{m-1}(2^{m-1} \pm 1). \end{aligned}$$

The vertex set of $VO_{2m}^\pm(2)$ is $\mathbb{F}_2^{2m}$, and adjacency is defined by $Q(x+y) = 0$, where $Q(x)$ is a nondegenerate quadratic form of type ± 1 . $Q(x)$ linearizes to the symplectic bilinear form

$$\langle x, y \rangle = Q(x+y) + Q(x) + Q(y).$$

The complement graph $\overline{VO_{2m}^\pm(2)}$ has parameters

$$(2^{2m}, 2^{m-1}(2^m \mp 1), 2^{m-1}(2^{m-1} \mp 1), 2^{m-1}(2^{m-1} \mp 1)),$$

and can be given by the cochain

$$f(x, y) = Q(x+y) = Q(x) + Q(y) + \langle x, y \rangle.$$

Applying the coboundary operator, we get the 2-cocycle

$$\partial f(x, y, z) = \langle x, y \rangle + \langle x, z \rangle + \langle y, z \rangle$$

of the associated two-graph $\mathcal{T}$. The descendant $\mathcal{T}_0$ is the graph on the set of vertices $\mathbb{F}_2^{2m}$, where x and y are adjacent whenever $\langle x, y \rangle = 1$. By deleting the isolated vertex 0 of $\mathcal{T}_0$, we obtain the complement of the symplectic graph over $\mathbb{F}_2$, which is strongly regular with parameters $(2^{2m} - 1, 2^{2m-1}, 2^{2m-2}, 2^{2m-2})$; see [4, Section 2.5].

The automorphism group of the two-graph $\mathcal{T}$, that is, the stabilizer of ∂f is the semidirect product $G = \mathbb{F}_2^{2m} \rtimes Sp(2m, 2)$ with its 2-transitive action on $\mathbb{F}_2^{2m}$. The automorphism group of the affine polar graph $VO_{2m}^\pm(2)$ is the semidirect product $H = \mathbb{F}_2^{2m} \rtimes O^\pm(2m, 2)$. H is a transitive maximal subgroup of G .

5. LINEAR AND AFFINE ACTIONS ON THE SWITCHING CLASS

Let W be a vector space over a field K . An *affine combination* of $w_0, \dots, w_m \in W$ is $c_0 w_0 + \dots + c_m w_m$, where $c_0, \dots, c_m \in K$ with $c_0 + \dots + c_m = 1$. The set of all affine combinations of $w_0, \dots, w_m$ is the *affine subspace* W' spanned by $w_0, \dots, w_m$. The size of the smallest affine generating set is the *affine dimension* of W' . Any affine subspace is obtained as a translate of a linear subspace. The set of affine subspaces is preserved by all affine linear maps $x \mapsto xA + b$, where $b \in W$ and $x \mapsto xA$ is linear. Affine maps may be represented by matrices of the form

$$(1) \quad \begin{bmatrix} A & 0 \\ b & 1 \end{bmatrix}.$$

This representation allows us to compute fixed points and invariant subspaces of affine linear maps using linear algebra tools.

Proposition 5.1. *Let τ be 2-coboundary, $\mathcal{T} = (V, T)$ the associated two-graph, and $G = \text{Aut}(\mathcal{T})$. Fix a 1-cocycle $s \in V^{(1,*)}$ such that $\partial s = \tau$. Define*

$$\mathcal{W}(\tau) = \{h \in V^{(1,*)} \mid \partial h = \tau\}.$$

For $v \in V$, write

$$\mathcal{V}(v) = \{f \in V^{(0,*)} \mid f(v) = 0\}.$$

For $g \in G$ and $f \in \mathcal{V}(v)$, define

$$(2) \quad f^{[g]}(x) = f(x^{g^{-1}}) + f(v^{g^{-1}}),$$

$$(3) \quad f^{\{g\}}(x) = f(x^{g^{-1}}) + f(v^{g^{-1}}) + s(x^{g^{-1}}, v^{g^{-1}}) + s(x, v).$$

Then:

- (i) $f^{[g]}$ is a linear action of G on $\mathcal{V}(v)$;
- (ii) $f^{\{g\}}$ is an affine linear action of G on $\mathcal{V}(v)$;
- (iii) the map $u : f \rightarrow \partial f + s$ is an isomorphism between the G -sets $\mathcal{V}(v)$ and $\mathcal{W}(\tau)$.

Proof. (i) is straightforward. By $\partial s = \tau = \tau^g = \partial s^g$, we have

$$(4) \quad s(x, y) + s(x, v) + s(y, v) = s^g(x, y) + s^g(x, v) + s^g(y, v)$$

for all $x, y \in V$. We obtain (ii) from

$$\begin{aligned} (f^{\{g\}})^{\{h\}}(x) &= f^{\{g\}}(x^{h^{-1}}) + f^{\{g\}}(v^{h^{-1}}) + s(x^{h^{-1}}, v^{h^{-1}}) + s(x, v) \\ &= f(x^{h^{-1}g^{-1}}) + f(v^{g^{-1}}) + s(x^{h^{-1}g^{-1}}, v^{g^{-1}}) + s(x^{h^{-1}}, v) \\ &\quad + f(v^{h^{-1}g^{-1}}) + f(v^{g^{-1}}) + s(v^{h^{-1}g^{-1}}, v^{g^{-1}}) + s(v^{h^{-1}}, v) \\ &\quad + s(x^{h^{-1}}, v^{h^{-1}}) + s(x, v) \\ &= f(x^{h^{-1}g^{-1}}) + f(v^{h^{-1}g^{-1}}) + s^g(x^{h^{-1}}, v) + s^g(v^{h^{-1}}, v) \\ &\quad + s(x^{h^{-1}}, v) + s(x^{h^{-1}}, v^{h^{-1}}) + s(v^{h^{-1}}, v) + s(x, v) \\ &\stackrel{(4)}{=} f(x^{h^{-1}g^{-1}}) + f(v^{h^{-1}g^{-1}}) + s^g(x^{h^{-1}}, v^{h^{-1}}) + s(x, v) \\ &= f(x^{(gh)^{-1}}) + f(v^{(gh)^{-1}}) + s(x^{(gh)^{-1}}, v^{(gh)^{-1}}) + s(x, v) \\ &= f^{\{gh\}}(x). \end{aligned}$$

We show (iii). Since $\partial(u(f)) = \partial s = \tau$, u maps $\mathcal{V}(v)$ to $\mathcal{W}(\tau)$. If $h \in \mathcal{W}(\tau)$, then $\partial(h - s) = 0$, and $h - s = \partial f$ with $f \in V^{(0,*)}$. Replacing f with $f + f(v)$, we can assume that $f(v) = 0$; therefore, u is surjective. If $u(f_1) = u(f_2)$, then $\partial(f_1 - f_2) = 0$, which implies $f_1 = f_2$ by $f_1(v) = f_2(v) = 0$. It remains to show that u is an isomorphism between G -sets. Fix $f \in \mathcal{V}(v)$, $g \in G$ and define

$$\varphi(x) = f^{\{g\}}(x) = f^g(x) + f^g(v) + s^g(x, v) + s(x, v).$$

Then,

$$\begin{aligned} (\partial\varphi + s)(x, y) &= f^g(x) + f^g(y) + s^g(x, v) + s^g(y, v) + s(x, v) + s(y, v) + s(x, y) \\ &\stackrel{(4)}{=} f^g(x) + f^g(y) + s^g(x, y) \\ &= (\partial f + s)^g(x, y). \end{aligned}$$

This proves $u(f^{\{g\}}) = u(f)^g$. □

We will call the actions (2) and (3) the *square bracket action* and the *curly bracket action*. When the element $v \in V$ is fixed, the natural choice for the 1-cocycle s is the

contraction $s(x, y) = \Delta_v \tau(x, y) = \tau(x, y, v)$. In this case, $s(x, v) = 0$, $s(x^{g^{-1}}, v^{g^{-1}}) = \tau(x^{g^{-1}}, v^{g^{-1}}, v) = \tau^g(x, v, v^g) = \tau(x, v, v^g)$. The curly bracket action is

$$(5) \quad f^{\{g\}}(x) = f^g(x) + f^g(v) + \tau(x, v, v^g).$$

We finish this section with a useful invariant of the curly bracket action. Define the sign of the cocycle $f \in \mathcal{V}(v)$ by

$$(6) \quad \text{sgn}(f) = \sum_{x \in V} f(x).$$

Then, for any $g \in G$,

$$\begin{aligned} \text{sgn}(f^{\{g\}}) &= \sum_{x \in V} f^{\{g\}}(x) \\ &= \sum_{x \in V} f(x^{g^{-1}}) + \sum_{x \in V} f(v^{g^{-1}}) \\ &= \sum_{x \in V} f(x^{g^{-1}}) \\ &= \text{sgn}(f). \end{aligned}$$

Clearly, $\text{sgn} : \mathcal{V}(v) \rightarrow \mathbb{F}_2$ is an additive homomorphism.

6. LINEAR TYPE TWO-GRAPHS AND THEIR GRAPHS

In this section, we investigate the linear-type two-graphs $\mathcal{T}$ whose automorphism group is $P\Sigma L(2, q)$ for $q \equiv 1 \pmod{4}$, and we classify the vertex-transitive strongly regular graphs that arise in their switching classes. We show that, in the switching class of $\mathcal{T}$, there is exactly one complementary pair of strongly regular graphs with a transitive automorphism group, and that this pair matches the linear-type row of Table 1 exactly, including the stated conditions on q . Our argument proceeds in three stages: first, we determine the feasible parameter sets; next, we classify the transitive maximal subgroups H of $P\Sigma L(2, q)$ (Section 6.2); finally, we realize the graphs as cocycles invariant under the curly bracket action of H (Section 6.3).

We stress that strongly regular graphs belonging to the class of $\mathcal{T}$ (and thus sharing the same parameters) are already known to exist; a brief description of a known construction is given in Section 6.1. Nevertheless, this construction does not produce vertex-transitive graphs; to our knowledge, our vertex-transitive SRGs appear to be new and have not previously been documented in the literature.

Let p be a prime, f a positive integer, and $q = p^f$. We represent the projective general linear group $PGL(2, q)$ as the group of fractional linear maps acting on $V = \mathbb{F}_q \cup \{\infty\}$. For $a, b, c, d \in \mathbb{F}_q$, $ad - bc \neq 0$, we denote by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the fractional linear mapping

$$z \mapsto z \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{az + b}{cz + d}.$$

The projective special linear group $PSL(2, q)$ consists of the maps $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \in S_q$, where S_q is the set of nonzero squares in $\mathbb{F}_q$. The field automorphisms $V \mapsto x^{p^i}$ act on V as well. The groups $PTL(2, q)$ and $P\Sigma L(2, q)$ are the closure groups of $PGL(2, q)$ and $PSL(2, q)$ with the group of field automorphisms. If q is even, then $PGL(2, q) = PSL(2, q)$, while for odd q , $PSL(2, q)$ is a normal subgroup of index 2 in $PGL(2, q)$.

The projective general linear group $PGL(2, q)$ acts 3-transitively on V ; hence, it also acts transitively on the set of unordered triples of V . If q is odd, then both $PSL(2, q)$ and $P\Sigma L(2, q)$ act 2-transitively on V . If $q \equiv 3 \pmod{4}$, then $PSL(2, q)$ acts transitively on the set of unordered triples of V ; this explains the fact that in this case, there is no underlying two-graph. If $q \equiv 1 \pmod{4}$, then both $PSL(2, q)$ and $P\Sigma L(2, q)$ have two orbits T and T' in their action on unordered triples, and the two orbits are swapped by

$PGL(2, q)$. This means that (V, T) and (V, T') are isomorphic regular two-graphs with parameters $|V| = q + 1$ and regularity $a = \frac{1}{2}(q - 1)$. For the remainder of this section, we assume that $q \equiv 1 \pmod{4}$, $G = P\Omega L(2, q)$, and $\{0, 1, \infty\} \in T$. We use the notation $\mathcal{T} = (V, T)$ for the regular two-graph of linear type.

Fix a triple $Z = \{x, y, \infty\} \in T$. As $|PSL(2, q)| = 3|T|$, the setwise stabilizer of Z in G acts transitively on Z . Swapping x and y if needed, we find an element $g \in G$ with $0^g = x$, $1^g = y$, and $\infty^g = \infty$. These conditions imply that g has the form $g : z \mapsto (y - x)z^{p^i} + x$ with $y - x \in S_q$. Conversely, if $y - x \in S_q$, then $g \in G$ exists and $\{x, y, \infty\} \in T$. This shows that the descendant $\mathcal{T}_\infty$ is the Paley graph $Paley(q)$ plus the isolated vertex ∞ .

6.1. Parameters of linear type SRGs. Let us assume that Γ is a strongly regular graph in the switching class of $\mathcal{T}$. Result 2.6 implies that Γ has parameters

$$(7) \quad \left(q + 1, \frac{q \pm \sqrt{q}}{2}, \frac{(\sqrt{q} \pm 1)^2}{4} - 1, \frac{(\sqrt{q} \pm 1)^2}{4} \right).$$

In particular, q is a square. For any odd square prime power q , a strongly regular graph in the switching class of $\mathcal{T}$ can be constructed in the following way. (The method seems to be folklore among experts.)

First, identify $\mathbb{F}_q$ with the affine plane $AG(2, \sqrt{q})$. Then the edges of $Paley(q)$ can be represented by the relation “the slope of the line connecting two points is zero or a square in $\mathbb{F}_{\sqrt{q}}$ ”.

- (1) Let Y be the union of $(\sqrt{q} \pm 1)/2$ parallel lines.
- (2) Add an isolated vertex ∞ to $Paley(q)$.
- (3) Switch with respect to Y .

The resulting graph $P(q, Y)$ is strongly regular with parameters as in (7). Moreover, there is a translation group A of order $\sqrt{q}$ that preserves the parallel class of Y . Since A fixes ∞ and $Paley(q)$, we have $A \leq \text{Aut}(P(q, Y))$.

In this paper, our focus is on strongly regular graphs with transitive automorphism groups. Result 6.2 of the next section classifies the transitive subgroups of G . It follows that transitive subgroups of G cannot have subgroups of order $\sqrt{q}$. In particular, the automorphism group of $P(q, Y)$ cannot act transitively on V .

6.2. Transitive subgroups of $P\Omega L(2, q)$. Even though the subgroup structure of $PSL(2, q)$ has been known since the beginning of the 20th century, the existence of transitive subgroups of $P\Omega L(2, q)$ is somewhat complicated. Our primary reference is Giudici [11].

Fix a nonsquare element $s \in \mathbb{F}_q$. Then

$$\begin{aligned} C_{q-1} &= \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \mid a \in \mathbb{F}_q^* \right\}, \\ C_{q+1} &= \left\{ \begin{bmatrix} a & b \\ sb & a \end{bmatrix} \mid a, b \in \mathbb{F}_q, (a, b) \neq (0, 0) \right\} \end{aligned}$$

are cyclic subgroups of order $q - 1$ and $q + 1$ of $PGL(2, q)$. C_{q-1} has two fixed points $0, \infty$, and C_{q+1} acts regularly on V . Both of these groups have a unique subgroup of index 2:

$$\begin{aligned} C_{\frac{q-1}{2}} &= \left\{ \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} \mid a \in S_q \right\}, \\ C_{\frac{q+1}{2}} &= \left\{ \begin{bmatrix} a & b \\ sb & a \end{bmatrix} \mid a^2 - sb^2 \in S_q \right\}. \end{aligned}$$

Their normalizers in $PSL(2, q)$ are the dihedral groups

$$D_{q-1} = C_{\frac{q-1}{2}} \rtimes \left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle, \quad D_{q+1} = C_{\frac{q+1}{2}} \rtimes \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\rangle.$$

Lemma 6.1. *Let $G = P\Sigma L(2, q)$ for $q = p^f \equiv 1 \pmod{4}$, p an odd prime, and $f \geq 2$. Let $\phi : z \mapsto z^p$ denote the Frobenius automorphism of $\mathbb{F}_q$ and define the semilinear fractional maps*

$$\alpha = \phi \left[\begin{smallmatrix} 1 & 0 \\ 0 & s^{\frac{p-1}{2}} \end{smallmatrix} \right] : z \mapsto \frac{z^p}{s^{\frac{p-1}{2}}}$$

$$\beta = \phi \left[\begin{smallmatrix} 0 & 1 \\ s^{\frac{p+1}{2}} & 0 \end{smallmatrix} \right] : z \mapsto \frac{s^{\frac{p+1}{2}}}{z^p}.$$

Then,

$$N_G(D_{q+1}) = \begin{cases} D_{q+1}\langle\alpha\rangle = C_{\frac{q+1}{2}} \rtimes \langle\alpha\rangle & \text{if } p \equiv 1 \pmod{4}, \\ D_{q+1}\langle\beta\rangle = C_{\frac{q+1}{2}} \rtimes \langle\beta\rangle & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

In both cases, $N_G(D_{q+1}) \cong C_{\frac{q+1}{2}} \rtimes C_{2f}$.

Proof. The following properties are straightforward to verify:

- (1) $\alpha \in G \Leftrightarrow p \equiv 1 \pmod{4}$.
- (2) $\beta \in G \Leftrightarrow p \equiv 3 \pmod{4}$.
- (3) Both α and β normalize D_{q+1} .
- (4) For any integer k , $\alpha^k = \phi^k \left[\begin{smallmatrix} 1 & 0 \\ 0 & s^{\frac{k-1}{2}} \end{smallmatrix} \right]$. In particular, $\alpha^f = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right] \in D_{q+1}$.
- (5) $\alpha^2 = \beta^2$.

If $p \equiv 3 \pmod{4}$, then f is even by $q \equiv 1 \pmod{4}$. This completes the proof. $\square$

Result 6.2 ([11, Theorem 1.3]). *Let $G = P\Sigma L(2, q)$ for $q = p^f$, p an odd prime and $f \geq 2$. Then the maximal subgroups of G which do not contain $PSL(2, q)$ are:*

- (1) *the stabilizer of a point of the projective line,*
- (2) $N_G(D_{q-1})$ *for $q \neq 9$,*
- (3) $N_G(D_{q+1})$ *for $q \neq 9$,*
- (4) S_5 *for $p \equiv \pm 3 \pmod{10}$ and $f = 2$,*
- (5) $N_G(PSL(2, q_0))$ *with $q = q_0^r$ for some prime r (2 conjugacy classes if $r = 2$).*

The following lemma specifies the transitive maximal subgroups of $P\Sigma L(2, q)$ when $q \equiv 1 \pmod{4}$, that is, in the case of our interest.

Lemma 6.3. *Let $G = P\Sigma L(2, q)$ for $q = p^f$, $q \equiv 1 \pmod{4}$, p prime and $f \geq 2$.*

- (i) *If $p \equiv 1 \pmod{4}$, then all transitive subgroups of G contain $PSL(2, q)$.*
- (ii) *If $q = 9$, then there are 2 conjugacy classes of transitive maximal subgroups $H \cong S_5$ of G not containing $PSL(2, q)$.*
- (iii) *If $q > 9$, $p \equiv 3 \pmod{4}$, then there is a unique conjugacy class of maximal transitive subgroups $H = N_G(D_{q+1})$ of G not containing $PSL(2, q)$.*

Proof. The stabilizer of a point is not a transitive subgroup. $N_G(D_{q-1})$ has a 2-point orbit $\{0, \infty\}$, and if $q = q_0^r$, $r > 1$, then $N_G(PSL(2, q_0))$ has an orbit of length $q_0 + 1$. By $f \geq 2$, S_5 acts transitively on V if and only if $q = 9$. In this case, $N_G(D_{q+1})$ is contained in S_5 , hence not maximal. The two conjugacy classes of S_5 merge in $P\Gamma L(2, q)$. This proves (ii).

According to Result 6.2, it remains the case $q > 9$, $H = N_G(D_{q+1})$. Let P_0, P_∞ denote the $C_{\frac{q+1}{2}}$ -orbits of 0 and ∞ , respectively. We have

$$P_0 = \left\{ \frac{b}{a} \in \mathbb{F}_q \mid a^2 - sb^2 = c^2 \text{ for some } c \in \mathbb{F}_q \cup \{\infty\} \right\}.$$

This implies $\infty \notin P_0$, and V is the disjoint union of P_0 and P_∞ . As $0 \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right] = 0$, P_0 and P_∞ are D_{q+1} -orbits as well. Using the maps α, β of Lemma 6.1, we have

$$0^\alpha = 0 \quad \text{and} \quad 0^\beta = \infty.$$

This means that if $p \equiv 1 \pmod{4}$, then $H = D_{q+1}\langle\alpha\rangle$ preserves the orbit P_0 , hence H is not transitive. This proves (i). If $p \equiv 3 \pmod{4}$, then $H = D_{q+1}\langle\beta\rangle$ swaps the orbits P_0, P_∞ , that is, H is transitive on V , and (iii) holds. $\square$

6.3. Linear type SRGs with a transitive automorphism group.

Theorem 6.4. *Let p be a prime such that $p \equiv 3 \pmod{4}$, $q = p^{2e} > 9$, $G = P\Sigma L(2, q)$, and $\mathcal{T}$ the linear two-graph with automorphism group G . Then there is (up to isomorphism) a unique pair $\Gamma, \bar{\Gamma}$ of complementary strongly regular graphs in the switching class of $\mathcal{T}$, admitting a transitive automorphism group.*

Proof. We will use the terminology and notation of Sections 5 and 6.2. Let $C_{\frac{q+1}{2}} = \langle g \rangle$ and assume that $0 \neq f_0 \in \mathcal{V}(\infty)$ is fixed by g in the square bracket action. Then for all $x \in V$,

$$f_0(x) = f_0^{[g^2]}(x) = f_0(x^{g^{-2}}) + f_0^{[g]}(v^{g^{-1}}) + f_0(v^{g^{-1}}) = f_0(x^{g^{-2}}).$$

As g has odd order, this implies that f_0 is constant on the $C_{\frac{q+1}{2}}$ -orbits P_0, P_∞ . Since $f_0 \neq 0$ and $\infty \in P_\infty$, f_0 is uniquely defined:

$$f_0(x) = \begin{cases} 1 & \text{if } x \in P_0, \\ 0 & \text{if } x \in P_\infty. \end{cases}$$

Moreover, $|P_0| = (q+1)/2$ is odd and $\text{sgn}(f_0) = 1$.

Let $R \subseteq \mathcal{V}(\infty)$ be a $C_{\frac{q+1}{2}}$ -orbit in the curly bracket action, and define $f_1 = \sum_{f \in R} f$. Since the curly bracket action is affine, and R has an odd number of elements, f_1 is fixed by $C_{\frac{q+1}{2}}$ in the curly bracket action. Let $f_2 \in \mathcal{V}(\infty)$ be another cocycle, that is fixed by $C_{\frac{q+1}{2}}$ in the curly bracket action. Then $f_2 - f_1 \neq 0$ is $C_{\frac{q+1}{2}}$ -fixed in the square bracket action, hence $f_2 = f_0 + f_1$. In particular, in the curly bracket action, $C_{\frac{q+1}{2}}$ has exactly two fixed cocycles.

The subgroup $H = N_G(C_{\frac{q+1}{2}}) \cong C_{\frac{q+1}{2}} \rtimes C_{4e}$ is a maximal transitive subgroup of G . As $C_{\frac{q+1}{2}} \triangleleft H$, H preserves $\{f_1, f_2\}$ in the curly bracket action. By $\text{sgn}(f_0) = 1$, we have $\text{sgn}(f_1) \neq \text{sgn}(f_2)$, and the invariance of the sign implies that both f_1 and f_2 are fixed by H in the curly bracket action. Proposition 5.1(iii) implies that $\partial f_1 + \tau, \partial f_2 + \tau$ yield H -invariant strongly regular graphs Γ_1, Γ_2 in the switching class of $\mathcal{T}$. The two-graph of the complement $\bar{\Gamma}_1$ is the complement $\bar{\mathcal{T}}$. For any $\gamma \in PGL(2, q) \setminus PSL(2, q)$, one has $\bar{\mathcal{T}}^\gamma = \mathcal{T}$. Hence, $\bar{\Gamma}_1^\gamma$ is a strongly regular graph in the switching class of $\mathcal{T}$. Since Γ_1 is not isomorphic to its complement, we must have $\bar{\Gamma}_1 \cong \bar{\Gamma}_1^\gamma \cong \Gamma_2$.

Finally, let Δ be a strongly regular graph in the switching class of $\mathcal{T}$, admitting a transitive automorphism group $A = \text{Aut}(\Delta)$. Since A is contained in a transitive maximal subgroup of G , we have $C_{\frac{q+1}{2}} \leq A \leq N_G(C_{\frac{q+1}{2}})$ up to conjugacy in G . This implies that the cocycle of Δ is either $\partial f_1 + \tau$ or $\partial f_2 + \tau$, and $\Delta = \Gamma_1$ or $\Delta = \Gamma_2$. $\square$

7. NONEXISTENCE RESULTS

Recall that our aim is to classify all triples (G, H, Γ) , where G is a finite 2-transitive permutation group, H is a maximal transitive subgroup of G , Γ is a primitive strongly regular graph, $H = \text{Aut}(\Gamma)$, and G is the automorphism group of the associated two-graph of Γ . (To be more precise, we also require that H does not contain the maximal perfect subgroup of G .) Taylor's theorem [23] restricts the possibilities of G to the groups $ASp(2m, 2)$, $Sp(2m, 2)$, $PTU(3, q)$, $\text{Ree}(q)$, $P\Sigma L(2, q)$, HS , and Co_3 . In this section, we prove the following nonexistence results:

Theorem 7.1. *Let the triple (G, H, Γ) be as in Theorem 1.1. Then the following cases are not possible.*

- (i) $q > 5$, $PSU(3, q) \triangleleft G$.

- (ii) $q = 3^{2e+1}$, $e > 1$, $\text{Ree}(q) \triangleleft G$.
- (iii) $\text{Co}_3 \triangleleft G$.
- (iv) $G = \text{PSp}(2m, 2)$, $ab = m$, $b > 1$ odd, $H = \text{Sp}(2a, 2^b) \rtimes C_b$.
- (v) $G = \text{PSp}(2m, 2)$, $H = P_m$ is the stabilizer of a maximally totally isotropic subspace.
- (vi) m even, $G = \text{PSp}(2m, 2)$, $H = (\text{Sp}(m, 2) \times \text{Sp}(m, 2)) \rtimes C_2$.

Table 4 summarizes how the elimination carried out in this section relates to the classification of Table 1: for each type of 2-transitive two-graph, it lists the transitive maximal subgroups H of G , and indicates whether the corresponding triple (G, H, Γ) is ruled out by Theorem 7.1 or survives as a row of Table 1.

TABLE 4. Elimination of candidate triples (G, H, Γ) versus Table 1

Type	G, H	Outcome	Reference
Affine polar	$\text{ASp}(2m, 2)$, $H = \mathbb{F}_2^{2m} \rtimes O^\pm(2m, 2)$	survives	Table 1, row 1
Symplectic	$\text{Sp}(2m, 2)$, $H = O^\pm(2m, 2)$	survives	Table 1, row 2
	$\text{Sp}(2m, 2)$, $H = \text{Sp}(m, 4) \rtimes C_2$, m even	survives	Table 1, row 2
	$\text{Sp}(2m, 2)$, $H = \text{Sp}(2a, 2^b) \rtimes C_b$, $b > 1$ odd divisor of m	excluded	Thm. 7.1(iv)
	$\text{Sp}(2m, 2)$, $H = P_m$	excluded	Thm. 7.1(v)
	$\text{Sp}(2m, 2)$, $H = (\text{Sp}(m, 2) \times \text{Sp}(m, 2)) \rtimes C_2$, m even	excluded	Thm. 7.1(vi)
	$\text{Sp}(6, 2)$, $H = G_2(2)$	survives	Table 1, $U_3(3)$ -graph
	$\text{Sp}(8, 2)$, $H = S_{10}$	survives	Table 1, $J(10, 3, 1)$
	$\text{Sp}(8, 2)$, $H = \text{PSL}(2, 17)$	survives	Table 1, row 7
Linear	$P\Sigma L(2, q)$, $H = C_{\frac{q+1}{2}} \rtimes C_{4e}$ (or S_5 for $q = 9$)	survives	Table 1, row 3
Unitary	$\text{PTU}(3, q)$, $H = S_7$	survives ($q = 5$)	Table 1, Goethals graph
	$\text{PTU}(3, q)$	excluded ($q > 5$)	Thm. 7.1(i)
Ree	$\text{Ree}(q) \rtimes \text{Aut}(\mathbb{F}_q)$	excluded	Thm. 7.1(ii)
Sporadic	HS , $H = M_{22}$	survives	Table 1, M_{22} -graph
	Co_3	excluded	Thm. 7.1(iii)

Our primary tool is the seminal paper [16] by Liebeck, Praeger, and Saxl, in which they list all triples (G, A, B) , where G is an almost simple group with socle L , and A and B are maximal subgroups of G that do not contain L , and $G = AB$ holds. Here, $AB = \{ab \mid a \in A, b \in B\}$, and $G = AB$ is called a factorization of G . If G acts on V , and H is a transitive subgroup of G , then we have the factorization $G = G_v H$ for any $v \in V$. Moreover, if G acts primitively and H is maximal, then both factors are maximal subgroups. In [16], the authors split the classification of factorizations $G = AB$ of almost simple groups into the following subcases:

- (1) infinite families of factorizations with L classical ([16, Tables 1, 2 and 4]);
- (2) exceptional factorizations with L classical ([16, Table 3]);
- (3) factorizations of exceptional simple groups of Lie type ([16, Table 5]);
- (4) factorizations of sporadic groups ([16, Table 6]).

We focus on almost simple finite groups G acting 2-transitively on V , and we require the subgroup H not to contain the socle L of G . In the 2-transitive groups that are relevant from our point of view, the stabilizers are as follows:

G	G_v
$Sp(2m, 2)$	$O^\pm(2m, 2)$
$PTU(3, q)$	parabolic subgroup P_1 of order $q^3(q^2 - 1) \text{Aut}(\mathbb{F}_{q^2}) $
$Ree(q)$	parabolic subgroup P of order $q^3(q - 1) \text{Aut}(\mathbb{F}_q) $
$P\Sigma L(2, q)$	parabolic subgroup P_1 of order $\frac{1}{2}q(q - 1) \text{Aut}(\mathbb{F}_q) $
HS	$PSU(3, 5) \rtimes C_2$
Co_3	$McL \rtimes C_2$.

The following two lemmas summarize the relevant information on transitive subgroups of these 2-transitive groups. As we can see, the richest subgroup structure belongs to the symplectic group $PSp(2m, 2)$.

Lemma 7.2. *Let L be a finite simple group. Let G be a group such that $L \leq G \leq \text{Aut}(L)$. Assume that G acts 2-transitively on the set X .*

- (i) *If $G = HS$ is the Higman–Sims sporadic simple group, and $|X| = 176$, then G has a unique transitive maximal subgroup H ; $H \cong M_{22}$.*
- (ii) *If $G = PTU(3, 5) = PSU(3, 5) \rtimes S_3$, $|X| = 126$, and H is a transitive maximal subgroup of G that does not contain $L = PSU(3, 5)$, then $H \cong S_7$ is unique up to conjugacy.*
- (iii) *In the following cases, no proper subgroup of G acts transitively on X :*
 - (a) *$q > 5$ is an odd prime power, $L = PSU(3, q)$, and $|X| = q^3 + 1$.*
 - (b) *$q = 3^{2e+1}$, $e > 1$, L is the Ree group $Ree(q)$, and $|X| = q^3 + 1$.*
 - (c) *L is Conway's sporadic simple group Co_3 , and $|X| = 276$.*

Proof. (i) and (iii)(c) are in Table 6, and (ii) is an exceptional factorization given in Table 3 of [16]. (iii)(a) is in Table 1, and (iii)(b) follows from Theorem B of [16]. $\square$

Lemma 7.3. *Let $G = Sp(2m, 2)$ act 2-transitively on V^ε of size $2^{2m-1} + \varepsilon 2^{m-1}$, where $\varepsilon = \pm 1$. Let H be a transitive maximal subgroup of G . Then one of the following occurs:*

- (i) $H = O^{-\varepsilon}(2m, 2)$;
- (ii) $ab = m$, b prime, and $H = Sp(2a, 2^b) \rtimes C_b$;
- (iii) $\varepsilon = -1$, and $H = P_m$ is the stabilizer of a maximal totally isotropic subspace;
- (iv) $\varepsilon = -1$, m is even, and $H = (Sp(m, 2) \times Sp(m, 2)) \rtimes C_2$ is the stabilizer of a decomposition $\mathbb{F}_2^{2m} = U \oplus U^\perp$, where U is a totally nonisotropic subspace of dimension m ;
- (v) $m = 3$ and $H = G_2(2) \cong PSU(3, 3) \rtimes C_2$;
- (vi) $\varepsilon = -1$, $m = 4$, and $H = S_{10}$;
- (vii) $\varepsilon = +1$, $m = 4$, and $H = PSL(2, 17)$.

Proof. (i)–(iv) are infinite classes given in [16, Table 1]. [16, Table 2] tells us that for $q = 2^e$, $A = G_2(q)$, $B = O^\pm(6, q)$ is an infinite class of maximal factorizations $G = AB$ of $G = PSp(6, q)$. This gives (v) with $q = 2$ for both $\varepsilon = \pm 1$. (For $\varepsilon = -1$, $|V^-| = 28$, and $H = G_2(2) \cong PSU(3, 3) \rtimes C_2$ is doubly transitive.) (vi) and (vii) are exceptional factorizations given in [16, Table 4]. $\square$

7.1. Parameters of graphs and two-graphs of symplectic type. The rest of this section is devoted to showing that certain classes of transitive subgroups of $PSp(2m, 2)$ cannot act on strongly regular graphs admitting two-graphs of symplectic type.

Our notation is as usual: $m \geq 3$ is an integer, $\langle \cdot, \cdot \rangle$ is a nondegenerate symplectic bilinear form on $\mathbb{F}_2^{2m}$, and $G = Sp(2m, 2)$ is the symplectic group preserving $\langle \cdot, \cdot \rangle$. We denote by $V^\pm$ the set of quadratic forms of type ± 1 , linearizing to $\langle \cdot, \cdot \rangle$; G acts doubly transitively on both V^+ and V^- . Let $\mathcal{T}^\pm = (V^\pm, T^\pm)$ denote the two-graphs of symplectic type with $G = \text{Aut}(\mathcal{T}^\pm)$; their parameters are $v^\pm = |V^\pm| = 2^{2m-1} \pm 2^{m-1}$ and $a^\pm = 2^{2m-2} \pm 2^{m-1} - 2$; see Table 3. By Result 2.6, the parameters of $\mathcal{T}^\pm$ allow

for two sets of parameters of strongly regular graphs in the switching class of $\mathcal{T}^\pm$:

$$\begin{aligned} k_1^\pm &= 2^{2m-2} - 1, & \lambda_1^\pm &= 2^{2m-3} - 2, & \mu_1^\pm &= 2^{2m-3} \mp 2^{m-2}; \\ k_2^\pm &= 2^{2m-2} \pm 3 \cdot 2^{m-2} - 1, & \lambda_2^\pm &= 2^{2m-3} \pm 3 \cdot 2^{m-2} - 2, & \mu_2^\pm &= 2^{2m-3} \pm 2^{m-1}. \end{aligned}$$

The first quadruple $(v^\pm, k_1^\pm, \lambda_1^\pm, \mu_1^\pm)$ gives the parameters of $NO_{2m}^\mp(2)$. If m is even, then the second quadruple $(v^\pm, k_2^\pm, \lambda_2^\pm, \mu_2^\pm)$ gives the parameters of $NO_{m+1}^\pm(4)$. If m is odd, then, in general, it is not known if an $sg(v^\pm, k_2^\pm, \lambda_2^\pm, \mu_2^\pm)$ exists.

In both cases, the eigenvalues are $\mp 2^{m-2} - 1$ and $\pm 2^{m-1} - 1$. The corresponding multiplicities are

$$(8) \quad f_1^\pm = \frac{4}{3}(2^{2m-2} - 1), \quad g_1^\pm = \frac{1}{3}(2^{m-1} \pm 1)(2^m \pm 1),$$

$$(9) \quad f_2^\pm = f_1^\pm + 1 = \frac{1}{3}(2^{2m} - 1), \quad g_2^\pm = g_1^\pm - 1 = \frac{2}{3}(2^{m-2} \pm 1)(2^m \mp 1).$$

7.2. Field extension subgroup $H = Sp(2a, 2^b) \rtimes C_b$. If b is a positive divisor of m , then $H(b) = Sp(\frac{2m}{b}, 2^b) \rtimes C_b$ is a subgroup of $G = Sp(2m, 2)$ with the following properties:

- (1) $H(b)$ is transitive in both actions of G on $2^{2m-1} \pm 2^{m-1}$ points.
- (2) $H(b_1) \leq H(b_2)$ if and only if $b_2 \mid b_1$.
- (3) $H(b)$ is maximal in G if and only if b is prime.
- (4) $H(m) = Sp(2, 2^m) \cong PGL(2, 2^m)$.

$H(b)$ is also called a field extension subgroup of $PSp(2m, 2)$. In this section, we prove the following nonexistence result:

Proposition 7.4. *Assume that $m = ab$, $b > 1$ is odd, and $H = Sp(2a, 2^b) \rtimes C_b$. There is no H -invariant strongly regular graph in the switching class of $\mathcal{T}^\pm$.*

The cases when m is even or odd require different approaches; the odd case is harder. Our method relies on the irreducible characters of $PGL(2, q)$, q even. Jordan and Schur computed the character table of $PGL(2, q)$ in 1907, and it appears in many textbooks (see [15]). We use the notation and terminology of [17]. As above, we denote by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ the elements of $PGL(2, q)$. Table 5 shows the structure of the conjugacy classes; Table 6 is the character table of $PGL(2, q)$, q even. The maps $\beta : \mathbb{F}_q^* / \mathbb{F}_q^* \rightarrow \mathbb{C}$ and $\gamma : \mathbb{F}_q^* \rightarrow \mathbb{C}$ are nontrivial homomorphisms from the cyclic groups of order $q \pm 1$. Moreover, $\eta_\beta = \eta_{\beta'}$ and $\nu_\gamma = \nu_{\gamma'}$ if and only if $\beta'(x) = \beta(x^{-1})$ and $\gamma'(x) = \gamma(x^{-1})$. Therefore, the number of these characters is $\frac{q}{2}$ and $\frac{q}{2} - 1$, respectively.

TABLE 5. Conjugacy classes in $PGL(2, q)$, q even

Representative of the conjugacy class	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$d_x = \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix}$ $x \in \mathbb{F}_q \setminus \{0, 1\}$	$h_r = \begin{bmatrix} 0 & 1 \\ r^{q+1} & r+r^q \end{bmatrix}$ $r \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$
Condition of conjugacy			$x = y$ or $x = y^{-1}$	$r\mathbb{F}_q^* = s\mathbb{F}_q^*$ or $r\mathbb{F}_q^* = s^{-1}\mathbb{F}_q^*$
Number of such classes	1	1	$\frac{1}{2}q - 1$	$\frac{1}{2}q$
Size of the conjugacy class	1	$q^2 - 1$	$q(q+1)$	$q(q-1)$

Lemma 7.5. *Let $U^\pm$ be the maximal subgroup $D_{2(q \mp 1)}$ of $L = PGL(2, q)$, q even. Let $\pi^\pm = 1_{U^\pm}^L$ be the permutation characters of the action of L on the right cosets of $U^\pm$. Then*

$$\pi^- = \lambda_1 + \sum_{\gamma} \nu_{\gamma}, \quad \pi^+ = \lambda_1 + \psi_1 + \sum_{\gamma} \nu_{\gamma},$$

where the summation is over the nontrivial homomorphisms $\gamma : \mathbb{F}_q^* \rightarrow \mathbb{C}$ up to inversion.

TABLE 6. Character table of $PGL(2, q)$, q even

	1	u	d_x	h_r
λ_1	1	1	1	1
ψ_1	q	0	1	-1
η_β ($\beta : \mathbb{F}_{q^2}^* / \mathbb{F}_q^* \rightarrow \mathbb{C}$)	$q-1$	-1	0	$-\beta(r) - \beta(r^{-1})$
ν_γ ($\gamma : \mathbb{F}_q^* \rightarrow \mathbb{C}$)	$q+1$	1	$\gamma(x) + \gamma(x^{-1})$	0

TABLE 7. Values of the permutation characters

	1	u	d_x	h_r
π^-	$\frac{1}{2}q(q-1)$	$\frac{1}{2}q$	0	1
π^+	$\frac{1}{2}q(q+1)$	$\frac{1}{2}q$	1	0

Proof. Table 7 displays the values of the permutation characters $\pi^\pm$. We immediately have $\pi^+ - \pi^- = \psi_1$. Straightforward computation shows

$$\langle \pi^-, \lambda_1 \rangle = \langle \pi^-, \nu_\gamma \rangle = 1$$

for all nontrivial homomorphisms $\gamma : \mathbb{F}_q^* \rightarrow \mathbb{C}$. Since the number of characters ν_γ is $\frac{q}{2} - 1$, and their degree is $q+1$, we have $\pi^- = \lambda_1 + \sum_\gamma \nu_\gamma$. $\square$

Proposition 7.6. *Let q be a power of 2, and Γ a strongly regular graph on $v = \frac{1}{2}q(q \pm 1)$ vertices. Assume that Γ admits $PGL(2, q)$ as a transitive automorphism group. Let f, g be the multiplicities of the restricted eigenvalues of Γ .*

- (1) *If $v = \frac{1}{2}q(q-1)$, then both f, g are divisible by $q+1$.*
- (2) *If $v = \frac{1}{2}q(q+1)$, then exactly one of f, g is divisible by $q+1$.*

Proof. Let r, s be restricted eigenvalues of Γ , with eigenspaces R, S . Then R, S are $PGL(2, q)$ -modules, and we have the decomposition $\mathbb{C}^v = \langle \mathbf{j} \rangle \oplus R \oplus S$, where $\mathbf{j}$ is the all-one vector of dimension v . In particular, R and S do not have the trivial module as an irreducible summand. If $v = \frac{1}{2}q(q-1)$, then both characters χ_R, χ_S are the sum of irreducible characters of degree $q+1$ by Lemma 7.5. Therefore, $f = \dim(S)$ and $g = \dim(R)$ are divisible by $q+1$. If $v = \frac{1}{2}q(q+1)$, then exactly one of χ_R, χ_S has a q -dimensional irreducible summand, while the other has only irreducible summands of degree $q+1$. This proves the proposition. $\square$

We are now able to prove the nonexistence result for $H(b)$ when b is an odd integer.

Proof of Proposition 7.4. Let $b > 1$ be an odd divisor of the integer m , and Γ be a putative $H(b)$ -invariant strongly regular graph in the switching class of $\mathcal{T}^\pm$.

Case 1: m is odd. As we have seen at the beginning of the section, $H(m) = Sp(2, 2^m) \cong PGL(2, 2^m)$ is a transitive subgroup of $H(b)$. Hence, Proposition 7.6 applies for Γ . The possible multiplicities of the eigenvalues of Γ are given in (8) and (9). Clearly, $f_1^\pm = \frac{1}{3}(2^m - 2)(2^m + 2)$ is not divisible by $2^m + 1$, and

$$f_2^\pm = \frac{1}{3}(2^m - 1)(2^m + 1) \text{ is divisible by } 2^m + 1 \iff 3 \mid 2^m - 1 \iff m \text{ is even.}$$

This shows that Γ cannot be in the switching class of $\mathcal{T}^-$. Assume that Γ is in the switching class of $\mathcal{T}^+$. Then $g_2^+ = \frac{2}{3}(2^{m-2} + 1)(2^m - 1)$ is not divisible by $2^m + 1$, and

$$g_1^+ = \frac{1}{3}(2^{m-1} + 1)(2^m + 1) \text{ is divisible by } 2^m + 1 \iff 3 \mid 2^{m-1} + 1 \iff m \text{ is even.}$$

Hence, we have a contradiction to Proposition 7.6.

Case 2: m is even. In this case, the switching class of $\mathcal{T}^\pm$ contains the strongly regular graph $\Delta = NO_{m+1}^\pm(4)$, see Section 3.3. The automorphism group of Δ is

the maximal subgroup $H(2) = Sp(m, 4) \rtimes C_2$ of G . By definition, $2b$ divides m , and $H(m)$ is a subgroup of both $\text{Aut}(\Gamma)$ and $\text{Aut}(\Delta)$. If S denotes the switching set from Γ to Δ , then the partition $S \dot{\cup} \bar{S}$ is preserved by $H(m)$. This is not possible, since $H(m) = Sp(2, 2^m) \rtimes C_m \cong PGL(2, 2^m) \rtimes C_m$ acts primitively on V . $\square$

7.3. Stabilizer $H = P_m$ of a maximal totally isotropic subspace. We say that a subspace $U \leq \mathbb{F}_2^{2m}$ is totally isotropic if, for any two vectors $x, y \in U$, the bilinear form $\langle x, y \rangle = 0$ holds. Then $\dim(U) \leq m$, and if $\dim(U) = m$, then U is a maximal totally isotropic subspace. Maximal totally isotropic subspaces are in the same G -orbit. The stabilizer of a maximal totally isotropic subspace in G is the m th parabolic subgroup P_m of G . By Lemma 7.3(iii), P_m is a maximal subgroup acting transitively on V^- .

Proposition 7.7. *There is no P_m -invariant strongly regular graph in the switching class of $\mathcal{T}^-$.*

Proof. The main ingredient of the proof is a subgroup L of G with the following properties:

- (1) L is contained in a (conjugate of) P_m and $O^+(2m, 2)$.
- (2) L is simple.
- (3) L is transitive on V^- .

We first show that (1)-(3) imply the proposition. Assume the converse and let Γ be a strongly regular graph in the switching class of $\mathcal{T}^-$ with $L \leq \text{Aut}(\Gamma)$. Let $\Delta = NO_{2m}^-(2)$ be a symplectic strongly regular graph with $\text{Aut}(\Delta) = O^+(2m, 2)$. Also, Δ is in the switching class of $\mathcal{T}^-$, and there is a switching set S from Δ to Γ . Since $L \leq \text{Aut}(\Delta)$, L must preserve the partition $S \dot{\cup} \bar{S}$. By (3), this is only possible if $|S| = \frac{v^-}{2}$, and L permutes the two sets $S, \bar{S}$. This contradicts (2).

To construct L , we use the hyperbolic quadratic form

$$Q^+(x_1, \dots, x_{2m}) = x_1x_{m+1} + \dots + x_mx_{2m}.$$

Put

$$L = \left\{ \begin{bmatrix} A & 0 \\ 0 & A^{-t} \end{bmatrix} \mid A \in GL(m, 2) \right\}.$$

Clearly, $L \cong GL(m, 2)$ is simple, L preserves Q^+ , and L fixes the maximal totally isotropic subspace $x_1 = \dots = x_m = 0$. To show (3), we compute the stabilizer K of the elliptic quadratic form

$$Q^-(x_1, \dots, x_{2m}) = x_1^2 + x_1x_{m+1} + x_{m+1}^2 + x_2x_{m+2} + \dots + x_mx_{2m}$$

in L . The element $M = \begin{bmatrix} A & 0 \\ 0 & A^{-t} \end{bmatrix}$ preserves Q^- if and only if both A and A^t preserve

the hyperplane $x_1 = 0$ of $\mathbb{F}_2^m$. This happens if and only if $M = \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}$ with $B \in GL(m-1, 2)$. As $|GL(m, 2) : GL(m-1, 2)| = (2^m - 1)2^{m-1}$, we get that the L -orbit of Q^- has length $|V^-| = 2^{2m-1} - 2^{m-1}$, proving the transitivity of L on V^- . $\square$

7.4. Stabilizer $H = (Sp(m, 2) \times Sp(m, 2)) \rtimes C_2$ of a decomposition. The subspace $U \leq \mathbb{F}_2^{2m}$ is nondegenerate if the restriction of $\langle \cdot, \cdot \rangle$ to U is nondegenerate. Equivalently, $U \cap U^\perp = 0$. Clearly, $\dim(U) \leq m$, and equality is possible if and only if m is even. Assume now that this is the case, and let U be a nondegenerate subspace of dimension m . Then $\mathbb{F}_2^{2m}$ decomposes as $\mathbb{F}_2^{2m} = U \oplus U^\perp$. Let $S_{\{U, U^\perp\}}$ denote the stabilizer of this decomposition. Then $S_{\{U, U^\perp\}} \cong (Sp(m, 2) \times Sp(m, 2)) \rtimes C_2$. By Lemma 7.3(iv), $S_{\{U, U^\perp\}}$ is a maximal subgroup acting transitively on V^- .

Proposition 7.8. *Assume that m is even. There is no $S_{\{U, U^\perp\}}$ -invariant strongly regular graph in the switching class of $\mathcal{T}^-$.*

Proof. In this proof, we will use quadratic forms on $\mathbb{F}_2^{2m}$, U , and $U^\perp$. We always assume that these forms linearize to $\langle \cdot, \cdot \rangle$ or its restriction. Any quadratic form Q defines the forms $Q_1 = Q|_U$ and $Q_2 = Q|_{U^\perp}$. All vectors x decompose as $x = u_1 + u_2$ with $u_1 \in U$, $u_2 \in U^\perp$, and $Q(x) = Q_1(u_1) + Q_2(u_2)$. This implies that Q_1, Q_2 determine Q uniquely. Using the standard form of quadratic forms over finite fields, it is immediate that Q is elliptic whenever Q_1 and Q_2 have different types. Counting the pairs (Q_1, Q_2) of different type quadratic forms, we obtain a bijection between V^- and $P_1 \dot{\cup} P_2$, where P_1 is the set of pairs (Q_1, Q_2) with Q_1 an elliptic quadratic form on U , Q_2 a hyperbolic quadratic form on $U^\perp$, and P_2 is the set of pairs (Q_1, Q_2) with Q_1 a hyperbolic quadratic form on U , Q_2 an elliptic quadratic form on $U^\perp$. The group $S_{\{U, U^\perp\}}$ preserves the partition $P_1 \dot{\cup} P_2$ and acts transitively on P_1 and P_2 . Even more is true: fix a pair $(Q_1, Q_2) \in P_1$; it has a stabilizer $H = O^-(m, 2) \times O^+(m, 2)$ in $S_{\{U, U^\perp\}}$. As $O^-(m, 2)$ is transitive on the set of hyperbolic quadric forms on U , and $O^+(m, 2)$ is transitive on the set of elliptic quadratic forms on $U^\perp$, we obtain that H acts transitively on P_2 .

Let Γ be an $S_{\{U, U^\perp\}}$ -invariant strongly regular graph in the switching class of $\mathcal{T}^-$. By the last observation, any vertex in P_1 is connected to all or no vertices of P_2 . Since the group is transitive on both P_1 and P_2 , we have either no edges or all edges between them. Switching with respect to P_1 , we can assume that no vertex of P_1 is connected to a vertex in P_2 . Hence, Γ is not connected, a contradiction. $\square$

7.5. Proof of the main nonexistence result.

Proof of Theorem 7.1. Parts (i), (ii) and (iii) are in Lemma 7.2(iii). Parts (iv), (v) and (vi) are in Propositions 7.4, 7.7 and 7.8, respectively. $\square$

8. UNIQUENESS RESULTS

In this section, we deal with the question of the uniqueness of the strongly regular graph Γ when its associated two-graph $\mathcal{T}$, the automorphism group $G = \text{Aut}(\mathcal{T})$, and the transitive maximal subgroup $H \leq G$ are given. At the end of the section, we complete the proof of the main result Theorem 1.1.

8.1. Uniqueness of affine polar graphs. Let $m \geq 2$ be an integer, and $G = \text{ASp}(2m, 2)$ and $\mathcal{T}$ be the affine two-graph. When we plug in the two-graph parameters $v = 2^{2m}$, $a = 2^{2m-1} - 2$ from Table 3 into the formulas of Result 2.6(ii), we obtain the parameters

$$\begin{aligned} v &= 2^{2m}, & k &= 2^{2m-1} \pm 2^{m-1} - 1, \\ \lambda &= 2^{2m-2} \pm 2^{m-1} - 2, & \mu &= 2^{2m-2} \pm 2^{m-1} \end{aligned}$$

of the affine polar graphs $VO_{2m}^\pm(2)$. Recall that the vertex set of $VO_{2m}^\pm(2)$ is $\mathbb{F}_2^{2m}$, and adjacency is defined by $Q(x+y) = 0$, where $Q(x)$ is a nondegenerate quadratic form of type ± 1 ; see Section 4. We aim to show that these graphs are the only strongly regular graphs Γ in the switching class of $\mathcal{T}$ such that $\text{Aut}(\Gamma)$ is a transitive maximal subgroup of G . We first classify the transitive maximal subgroups of G .

Lemma 8.1. *Let $G = \text{ASp}(2m, 2)$ act 2-transitively on $V = \mathbb{F}_2^{2m}$. Let N be the regular normal subgroup of G , and $S = G_0 \cong \text{Sp}(2m, 2)$ the stabilizer of $0 \in V$. If H is a transitive maximal subgroup of G , then $N \leq H$, $H = NT$, where T is a maximal subgroup of S .*

Proof. Assume that $N \not\leq H$. As H is maximal, $G = NH$. $N \cap H$ is normal in both N and H , hence $N \cap H \triangleleft G$. This implies $N \cap H = 1$ because N is a minimal normal subgroup of G . Then $H \cong \text{Sp}(2m, 2)$, and the transitivity of H contradicts [13, Theorem 1]. $\square$

Proposition 8.2. *Let $m \geq 2$ be an integer, $G = \text{ASp}(2m, 2)$, and H a transitive maximal subgroup of G . Let $\mathcal{T}$ be the affine two-graph with automorphism group G , and Γ a strongly regular graph in the switching class of $\mathcal{T}$ with $\text{Aut}(\Gamma) = H$. Then for $\phi \in \{\pm 1\}$, $H \cong \mathbb{F}_2^{2m} \rtimes O^\phi(2m, 2)$, and $\Gamma \cong VO_{2m}^\phi(2)$.*

Proof. Choose ϵ such that Γ has parameters

$$(2^{2m}, 2^{2m-1} + \epsilon 2^{m-1} - 1, 2^{2m-2} + \epsilon 2^{m-1} - 2, 2^{2m-2} + \epsilon 2^{m-1}).$$

Having the same associated two-graph, we can switch Γ with a set S in the graph $VO_{2m}^{-\epsilon}(q)$. Also, we can assume $0 \in S$ without loss of generality. By Result 2.5(2), $|S| = 2^{2m-1}$.

By Lemma 8.1, both $\text{Aut}(\Gamma)$ and $\text{Aut}(VO_{2m}^{-\epsilon}(2))$ contain the minimal normal subgroup N of G . This means that N leaves the two edge sets $E(\Gamma)$, $E(VO_{2m}^{-\epsilon}(2))$ invariant. This implies that the symmetric difference of $E(\Gamma)$ and $E(VO_{2m}^{-\epsilon}(2))$ is N -invariant as well. Since S is a switching set, this implies that N preserves the partition $V = S \dot{\cup} \bar{S}$. Fix arbitrary elements $w, z \in S$. The translation $x \mapsto x + w$ is an element of N , which leaves S invariant by $0, w \in S$. Hence, $w + z \in S$, and S is an $\mathbb{F}_2$ -linear subspace. As $|S| = 2^{2m-1}$, S has codimension 1, and there is a linear functional $\alpha : V \rightarrow \mathbb{F}_2$ such that $S = \ker(\alpha)$.

Adjacency in $VO_{2m}^{-\epsilon}(2)$ is given by $Q(x + y) = 0$, where $Q(x)$ is a nondegenerate quadratic form of type $-\epsilon$. Define the quadratic form

$$Q^*(x) = Q(x) + \alpha(x)^2 = Q(x) + \alpha(x).$$

On the one hand, Q and Q^* polarize to the same symplectic form; hence, Q^* defines a strongly regular affine polar graph $\Delta \cong VO_{2m}^\phi(2)$ in the switching class of $\mathcal{T}$, $\phi = \pm 1$. On the other hand,

$$0 = Q^*(x + y) = Q(x + y) + \alpha(x) + \alpha(y)$$

if and only if either x, y are adjacent in $VO_{2m}^{-\epsilon}(q)$ and both are in S or both are in $\bar{S}$, or x, y are nonadjacent in $VO_{2m}^{-\epsilon}(q)$ and one is in S and the other in $\bar{S}$. In other words, $Q^*(x + y) = 0$ if and only if x and y are adjacent in Γ . This proves $\Gamma = VO_{2m}^\phi(q)$, and $H \cong \mathbb{F}_2^{2m} \rtimes O^\phi(2m, 2)$ follows. $\square$

8.2. Uniqueness in two-graphs of symplectic type. We already know four infinite classes of strongly regular graphs associated with symplectic two-graphs: $NO_{2m}^\pm(2)$ and $NO_{2m+1}^\pm(4)$. In this subsection, we prove that, besides these classes, precisely three further graphs exist. By Section 7, we have a relatively short list of transitive maximal subgroups of $\text{PSp}(2m, 2)$. To find the graphs that admit these subgroups as the automorphism groups, we will determine their rank and the orbit lengths of the stabilizers.

As in Section 7, $m \geq 3$ is an integer, $\langle \cdot, \cdot \rangle$ is a nondegenerate symplectic bilinear form on $\mathbb{F}_2^{2m}$, and $G = \text{Sp}(2m, 2)$ is the symplectic group preserving $\langle \cdot, \cdot \rangle$. We denote by $V^\pm$ the set of quadratic forms of type ± 1 , linearizing to $\langle \cdot, \cdot \rangle$; G acts doubly transitively on both V^+ and V^- .

Let H be a transitive maximal subgroup of G . By Theorem 7.1 and Lemma 7.3, one of the following holds:

- (A) $H = O^\pm(2m, 2)$.
- (B) m is even, and $H = \text{Sp}(m, 4) \rtimes C_2$.
- (C) $m = 3$, and $H = G_2(2) \cong \text{PSU}(3, 3) \rtimes C_2$.
- (D) $m = 4$, G acts on 120 points, and $H = S_{10}$.
- (E) $m = 4$, G acts on 136 points, and $H = \text{PSL}(2, 17)$.

The existence and uniqueness of a graph in the exceptional cases (C), (D), and (E), as well as the identification of their associated two-graphs, are discussed in Section 3; see

Proposition 3.1. We verify their uniqueness under the prescribed automorphism groups computationally. In what follows, we focus on the infinite classes in (A) and (B).

Lemma 8.3. *The subgroup $A = O^\pm(2m, 2)$ of $PSp(2m, 2)$ is transitive of rank 3 on $V^\mp$.*

Proof. Let $Q(x)$ be the quadratic form associated with $A = O^\epsilon(2m, 2)$. The action of A on $V^\mp$ is equivalent to its linear action on the set $\{x \mid Q(x) = 1\}$ of nonisotropic vectors. Take vectors x, y, x', y' with $x \neq y$, $x' \neq y'$, and $Q(x) = Q(y) = Q(x') = Q(y') = 1$. The map $x \mapsto x'$, $y \mapsto y'$ extends to an orthogonal linear map $g \in A$ if and only if $Q(x + y) = Q(x' + y')$. As $Q(x)$ can take only two values, the rank of the action is three. $\square$

Lemma 8.4. *Let $m = ab$, $q = 2^b$ and $\epsilon \in \{1, -1\}$.*

- (i) *For even m , the subgroup $B = Sp(2a, q)$ of $PSp(2m, 2)$ is transitive on both V^+ and V^- .*
- (ii) *For any element $x^\epsilon \in V^\epsilon$, the stabilizer B_{x^ϵ} is isomorphic to $O^\epsilon(2a, q)$.*
- (iii) *B_{x^ϵ} has one orbit of lengths 1, one orbit of length $(q^{a-1} + \epsilon)(q^a - \epsilon)$, and $(q - 1)$ orbits of length $q^{a-1}(q^a - \epsilon)$.*

Proof. (i) is Lemma 7.3(ii). Theorem 1.1(iii) of [18] tells us that $B_{x^\epsilon} \cong O^\epsilon(2a, q)$, and its action on $V^+ \cup V^-$ is equivalent to the linear action of $O^\epsilon(2a, q)$. This implies (ii). To see (iii), we introduce the notation $Q(x)$ for the quadratic form associated with $O^\epsilon(2a, q)$. The orbits of the linear action are $\{0\}$, $P_0 = \{x \neq 0 \mid Q(x) = 0\}$, and $P_\lambda = \{x \mid Q(x) = \lambda\}$. Since $cP_\lambda = P_{c^2\lambda}$, we have $|P_\lambda| = |P_\mu|$ for $\lambda, \mu \neq 0$.

In $PG(2a - 1, q)$, the equation $Q(x) = 0$ determines a nonsingular quadric $\mathcal{Q}^\epsilon$ of type ϵ . The number of projective points of $\mathcal{Q}^\epsilon$ is $(q^{2a-1} - 1)/(q - 1) + \epsilon q^{a-1}$. To a projective point of $\mathcal{Q}^\epsilon$ correspond precisely $q - 1$ vectors in P_0 , therefore,

$$|P_0| = (q - 1)|\mathcal{Q}^\epsilon| = q^{2a-1} - 1 + \epsilon q^{a-1}(q - 1) = (q^{a-1} + \epsilon)(q^a - \epsilon).$$

Since all P_λ , $\lambda \neq 0$ have the same cardinality x , we have

$$q^{2a} = 1 + (q^{a-1} + \epsilon)(q^a - \epsilon) + (q - 1)x.$$

Resolving for x , we obtain $|P_\lambda| = q^{a-1}(q^a - \epsilon)$, and (iii) follows. $\square$

Proposition 8.5. *Let Γ be a strongly regular graph on $v = 2^{2m-1} \pm 2^{m-1}$ vertices; $m \geq 3$. Assume that the associated two-graph of Γ is of symplectic type.*

- (i) *If $\text{Aut}(\Gamma) = O^\mp(2m, 2)$, then $\Gamma \cong NO_{2m}^\mp(2)$.*
- (ii) *If m is even and $\text{Aut}(\Gamma) = Sp(m, 4) \rtimes C_2$, then $\Gamma \cong NO_{m+1}^\pm(4)$.*

Proof. (i) is immediate by the rank 3 property of $O^\mp(2m, 2)$; see Lemma 8.3. (ii) Let us assume that m is even and $\text{Aut}(\Gamma) = Sp(m, 4) \rtimes C_2$. Let us plug in $a = m/2$, $q = 4$ in Lemma 8.4(iii). We obtain that the stabilizer of a vertex x has orbits of length 1, $2^{2m-2} \pm 3 \cdot 2^{m-2} - 1$, and $2^{m-2}(2^m \mp 1)$. However, Table 3 and Result 2.6 imply that the degree of Γ is either $2^{2m-2} - 1$ or $2^{2m-2} \pm 3 \cdot 2^{m-2} - 1$. The only way to express these numbers as linear combinations of the orbit lengths is when the degree is $k = 2^{2m-2} \pm 3 \cdot 2^{m-2} - 1$, and the neighbourhood of x is an orbit of the stabilizers. Hence, Γ is uniquely defined, and $\Gamma \cong NO_{m+1}^\pm(4)$. $\square$

8.3. Proof of the main theorem.

Proof of Theorem 1.1. Let Γ be a primitive strongly regular graph with associated two-graph $\mathcal{T}$, $H = \text{Aut}(\Gamma)$ transitive, and $G = \text{Aut}(\mathcal{T})$ doubly transitive. If $\mathcal{T}$ is of affine polar type, then $\Gamma = VO_{2m}^\pm(2)$ by Proposition 8.2. Let $\mathcal{T}$ be of linear type, and $G = P\Omega L(2, q)$. By (7), q is a square: $q = p^{2e}$ for some prime p . By Lemma 6.3(i), $p \equiv 3 \pmod{4}$. If $q = 9$, then Γ is the Petersen graph or its complement. If

$q > 9$, then Theorem 6.4 tells us that there are two choices for Γ , and these two are graph complements of each other. Moreover, their parameters differ, hence they are not isomorphic. When $\mathcal{T}$ is of unitary type, and $PSU(3, q) \leq G \leq P\Gamma U(3, q)$ then $q = 5$ by Theorem 7.1(i). For $q = 5$, we have $H = S_7$ by Lemma 7.2(ii), and Γ must be Goethals' S_7 -invariant $srg(126, 50, 13, 24)$. Theorem 7.1(ii) and (iii) exclude the cases when $\mathcal{T}$ is of Ree type, or sporadic type with $G = Co_3$. In the sporadic type two-graph with $G = HS$, Γ is uniquely determined by the parameters.

It remains to deal with the case of a symplectic type two-graph $\mathcal{T}$. Then, $G = PSp(2m, 2)$ with $m \geq 3$. The transitive maximal subgroups are given in Lemma 7.3, part of which is excluded by Theorem 7.1(iv)-(vi). The remaining exceptional cases are $(m, H) \in \{(3, G_2(2)), (4, S_{10}), (4, PSL(2, 17))\}$. The existence of the corresponding graphs and the identification of their associated two-graphs are given in Proposition 3.1. At the same time, uniqueness under the prescribed automorphism groups follows from the computations described in Subsection 8.2. The infinite classes of transitive maximal subgroups, and the existence and the uniqueness of the corresponding strongly regular graphs are given in Proposition 8.5. $\square$

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