On the metric dimension of imprimitive distance-regular graphs

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Abstract

A *resolving set* for a graph Γ is a collection of vertices S, chosen so that for each vertex v, the list of distances from v to the members of S uniquely specifies v. The *metric dimension* of Γ is the smallest size of a resolving set for Γ. Much attention has been paid to the metric dimension of distance-regular graphs. In this paper, we consider the metric dimension of three families of imprimitive distance-regular graphs: bipartite doubles, Taylor graphs, and the incidence graphs of symmeric designs. In each case, we demonstrate a connection between the metric dimension of Γ to that of a related primitive graph.

Keywords: metric dimension; resolving set; distance-regular graph; bipartite double; Taylor graph; incidence graph

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1 Introduction

A *resolving set* for a graph $\Gamma = (V,E)$ is a set of vertices $R = \{v_1,\ldots,v_k\}$ such that for each vertex $w \in V$, the list of distances $(\mathrm{d}(w,v_1),\ldots,\mathrm{d}(w,v_k))$ uniquely determines w. Equivalently, R is a resolving set for Γ if, for any pair of vertices $u,w \in V$, there exists $v_i \in R$ such that $\mathrm{d}(u,v_i) \neq \mathrm{d}(w,v_i)$; we say that v_i *resolves* u and w. The *metric dimension* of Γ is the smallest size of a resolving set for Γ. This concept was introduced to the graph theory literature in the 1970s by Harary and Melter [20] and, independently, Slater [25]; however, in the context of arbitrary metric spaces, the concept dates back at least as far as the 1950s (see Blumenthal [8], for instance). For further details, the reader is referred to the survey paper [4].

When studying metric dimension, distance-regular graphs are a natural class of graphs to consider. A graph Γ with diameter d is *distance-regular* if, for all i with $0 \leq i \leq d$ and any vertices u,v with $\mathrm{d}(u,v) = i$, the number of neighbours of v at distances $i-1$, i and $i+1$ from u depend only on the distance i, and not on the choices of u and v. These numbers are

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denoted by c_i, a_i and b_i respectively, and are known as the *parameters* of Γ. It is easy to see that c_0, b_d are undefined, $a_0 = 0$, $c_1 = 1$ and $c_i + a_i + b_i = k$ (where k is the valency of Γ). We put the parameters into an array, called the *intersection array* of Γ,

$$\left\{ \begin{array}{cccccc} * & 1 & c_2 & \cdots & c_{d-1} & c_d \\ 0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\ k & b_1 & b_2 & \cdots & b_{d-1} & * \end{array} \right\}.$$

In the case where Γ has diameter 2, we have a *strongly regular* graph, and the intersection array may be determined from the number of vertices n, valency k, and the parameters $a = a_1$ and $c = c_2$; in this case, we say (n,k,a,c) are the parameters of the strongly regular graph. Another important special case of distance-regular graphs are the *distance-transitive* graphs, i.e. those graphs Γ with the property that for any vertices u, v, u', v' such that $\mathrm{d}(u,v) = \mathrm{d}(u',v')$, there exists an automorphism g such that $u^g = u'$ and $v^g = v'$. For more information about distance-regular graphs, see the book of Brouwer, Cohen and Neumaier [9] and the forthcoming survey paper by van Dam, Koolen and Tanaka [13]. In recent years, a number of papers have been written on the subject of the metric dimension of distance-regular graphs (and on the related problem of class dimension of association schemes), by the present author and others: see [2, 3, 4, 5, 7, 14, 16, 17, 18, 19, 21], for instance. In this paper, we shall focus on various classes of *imprimitive* distance-regular graphs, which are explained below.

1.1 Imprimitive graphs

A distance regular graph Γ is *primitive* if and only if each of its distance-i graphs is connected, and is *imprimitive* otherwise. A result known as *Smith's Theorem* (after D. H. Smith, who proved it for the distance-transitive case [26]) states that there are two ways for a distance-regular graph to be imprimitive: either the graph is bipartite, or is *antipodal*. The latter case arises when the distance-d graph (where d is the diameter of Γ) consists of a disjoint union of cliques, so that the relation of being at maximum distance in Γ is an equivalence relation on the vertex set. The vertices of these cliques are referred to as *antipodal classes*; if the antipodal classes have size t, then we say Γ is t-*antipodal*. It is possible for a graph to be both bipartite and antipodal, with the hypercubes providing straightforward examples.

If Γ is a bipartite distance-regular graph, the distance-2 graph has two connected components; these components are the *halved graphs* of Γ. If Γ is antipodal, the *folded graph*, denoted $\overline{\Gamma}$, of Γ is defined as having the antipodal classes of Γ as vertices, with two classes being adjacent in $\overline{\Gamma}$ if and only if they contain adjacent vertices in Γ. The folded graph $\overline{\Gamma}$ is also known as an *antipodal quotient* of Γ; conversely, Γ is an *antipodal t-cover* of $\overline{\Gamma}$ (where t is the size of the antipodal classes). The operations of halving and folding may be used to reduce imprimitive graphs to primitive ones: see [9, §4.2] for details.

In this paper, we shall consider the metric dimension of three types of imprimitive distance-regular graphs: bipartite doubles, Taylor graphs, and the incidence graphs of symmetric designs. The first two types are both 2-antipodal, and we begin with a lemma about such graphs in general.

1.2 A lemma about antipodal 2-covers

In a 2-antipodal graph, each vertex has a unique antipode (i.e. vertex at maximum distance), so we may form a partition of the vertex set $V^+ \cup V^-$ so that for any vertex in V^+, the antipode is in V^-; for brevity, we refer to such a partition as a 2-*antipodal partition* for Γ. Note that a 2-antipodal graph has many such partitions (we may take any transversal of the antipodal classes, together with its complement), although in certain cases there natural partition arising from how the graph is constructed.

For any vertex v, we denote by $\Gamma_i(v)$ the set of vertices of Γ that are at distance i from v.

Lemma 1. *Suppose that Γ is a 2-antipodal distance regular graph of diameter d, whose vertex set has a 2-antipodal partition $V^+ \cup V^-$. Then, without loss of generality, a resolving set for $\widetilde{\Gamma}$ can be chosen just from vertices in V^+.*

Proof. We claim that if R is any resolving set for Γ and $v^- \in R$, then $(R \setminus \{v^-\}) \cup \{v^+\}$ is also a resolving set. To show this, suppose that x, y are resolved by v^-, i.e. $\mathrm{d}_\Gamma(x, v^-) \neq \mathrm{d}_\Gamma(y, v^-)$. Suppose that $d(x, v^+) = i$, i.e. $x \in \Gamma_i(v^+)$. Since Γ is distance-regular, there exists a path of length $d - i$ to some vertex in $\Gamma_d(v^+)$; however, as v^- is the unique vertex in $\Gamma_d(v^+)$, it follows that $\mathrm{d}_\Gamma(x, v^-) = d - i$, and so $d(x, v^+) + d(x, v^-) = d$. Therefore, $\mathrm{d}_\Gamma(x, v^+) = d - \mathrm{d}_\Gamma(x, v^-) \neq d - \mathrm{d}_\Gamma(y, v^-) = \mathrm{d}_\Gamma(y, v^+)$, and hence v^+ also resolves x, y.

By repeating the above process as required, an arbitrary resolving set for Γ may be transformed into a resolving set consisting only of vertices in V^+, and the result follows. $\square$

2 Bipartite doubles

Definition 2. Let $\Gamma = (V, E)$ be a graph. Then the *bipartite double* of Γ is the bipartite graph $\widetilde{\Gamma}$ whose vertex set consists of two disjoint copies of V, labelled V^+ and V^-, and where $u^+ \in V^+$ and $w^- \in V^-$ are adjacent in $\widetilde{\Gamma}$ if and only if u and w are adjacent in Γ.

If Γ is itself bipartite, then $\widetilde{\Gamma}$ consists of two disjoint copies of Γ, so we will assume otherwise. The bipartite double of a complete graph K_n is the graph $K_{n,n} - I$, i.e. a complete bipartite graph with a 1-factor removed. The bipartite double of the Petersen graph is known as the *Desargues graph*.

If Γ is distance-regular, then there are two situations where its bipartite double $\widetilde{\Gamma}$ will also be distance-regular.

Theorem 3 (cf. [9, Theorem 1.11.1])**.** *Suppose that Γ is distance-regular with diameter d. Then the bipartite double $\widetilde{\Gamma}$ is also distance-regular, provided that either:*

(i) *Γ has odd girth $2d+1$, whence $\widetilde{\Gamma}$ is distance-regular with diameter $2d+1$; or*

(ii) *Γ has diameter $d = 2j$, odd girth $2j+1$, and its parameters satisfy $a_j = c_{j+1}$ and $b_{j-1} = c_{j+1} + a_{j+1} = c_{j+i+1}$ (for each $i \in \{1, \ldots, j\}$), whence $\widetilde{\Gamma}$ is distance-regular with diameter $d+1$.*

In case (i), the bipartite double $\widetilde{\Gamma}$ is also an antipodal 2-cover of Γ; the bipartition $V^+ \cup V^-$ forms a 2-antipodal partition for $\widetilde{\Gamma}$. The most important example of case (ii) arises when Γ is strongly regular with parameters (n, k, a, a); in that case, the bipartite double is a bipartite distance regular graph of diameter 3, and is therefore the incidence graph of a symmetric design. Graphs arising in this way will be discussed later, in Section 4.

The proof of Theorem 3(i) relies upon the following lemma, which while left as an exercise to the reader, will be important in proving Theorem 5 below.

Lemma 4. *Suppose that Γ is a distance-regular graph of diameter d with odd girth $2d+1$. Then for any vertices u,v of Γ, and any vertex x of $\widetilde{\Gamma}$, we have:*

(i) $\mathrm{d}_{\widetilde{\Gamma}}(x,v^+)+\mathrm{d}_{\widetilde{\Gamma}}(x,v^-)=2d+1$*;*

(ii) $\mathrm{d}_{\widetilde{\Gamma}}(u^+,v^+)=\begin{cases} \mathrm{d}_\Gamma(u,v) & \text{if } \mathrm{d}_\Gamma(u,v) \text{ is even,} \\ 2d+1-\mathrm{d}_\Gamma(u,v) & \text{if } \mathrm{d}_\Gamma(u,v) \text{ is odd.}\end{cases}$

The main result of this section concerns the graphs in case (i) of Theorem 3, relating the metric dimension of Γ and $\widetilde{\Gamma}$.

Theorem 5. *Suppose that Γ is a distance-regular graph of diameter d with odd girth $2d+1$. Then the metric dimension of its bipartite double $\widetilde{\Gamma}$ is equal to the metric dimension of Γ.*

Proof. First, we recall from Lemma 1 that, since $V^+\cup V^-$ is a 2-antipodal partition for $\widetilde{\Gamma}$, we can assume without loss of generality that any resolving set for $\widetilde{\Gamma}$ is contained within V^+.

Second, we show that if R is a resolving set for Γ, then R^+ is a resolving set for $\widetilde{\Gamma}$. Suppose that R is a resolving set for Γ. If $x,y\in V^+$, let $x=u^+$ and $y=w^+$, for some $u,w\in V$. Then there exists $v\in R$ satisfying $\mathrm{d}_\Gamma(u,v)\neq \mathrm{d}_\Gamma(w,v)$. If these are both even, we have $\mathrm{d}_{\widetilde{\Gamma}}(u^+,v^+)=\mathrm{d}_\Gamma(u,v)\neq \mathrm{d}_\Gamma(w,v)=\mathrm{d}_{\widetilde{\Gamma}}(u^+,v^+)$. If these are both odd, we have $\mathrm{d}_{\widetilde{\Gamma}}(u^+,v^+)=2d+1-\mathrm{d}_\Gamma(u,v)\neq 2d+1-\mathrm{d}_\Gamma(w,v)=\mathrm{d}_{\widetilde{\Gamma}}(u^+,v^+)$. If these have different parities, then (without loss of generality) we have $0\leq \mathrm{d}_{\widetilde{\Gamma}}(u^+,v^+)\leq d$ and $d+1\leq \mathrm{d}_{\widetilde{\Gamma}}(w^+,v^+)\leq 2d+1$, and thus $\mathrm{d}_{\widetilde{\Gamma}}(u^+,v^+)\neq \mathrm{d}_{\widetilde{\Gamma}}(w^+,v^+)$. In all cases, it follows that v^+ resolves u^+,w^+. The same argument shows that, if $x,y\in V^-$, they will be resolved by a vertex $v^-\in R^-$; by applying the previous paragraph, they will also be resolved by the corresponding vertex $v^+\in R^+$. Also, we note that if $x\in V^+$ and $y\in V^-$, any vertex in V^+ will resolve them, as the distances will have different parities. Therefore R^+ is a resolving set for $\widetilde{\Gamma}$, and thus $\mu(\widetilde{\Gamma})\leq\mu(\Gamma)$.

Finally, we show the converse of the above. Suppose that R^+ is a resolving set for $\widetilde{\Gamma}$, with $R^+\subseteq V^+$. For all $u^+,w^+\in V^+$, there exists $v^+\in R^+$ with $\mathrm{d}_{\widetilde{\Gamma}}(u^+,v^+)\neq \mathrm{d}_{\widetilde{\Gamma}}(w^+,v^+)$. If these distances both lie in the interval $0,\ldots,d$ or are both in the interval $d+1,\ldots,2d+1$, then clearly $\mathrm{d}_\Gamma(u,v)\neq \mathrm{d}_\Gamma(w,v)$. Otherwise, we have (without loss of generality) that $0\leq \mathrm{d}_{\widetilde{\Gamma}}(u^+,v^+)\leq d$ and $d+1\leq \mathrm{d}_{\widetilde{\Gamma}}(w^+,v^+)\leq 2d+1$, which implies that $\mathrm{d}_\Gamma(u,v)$ is even and $\mathrm{d}_\Gamma(w,v)$ is odd. Therefore, R is a resolving set for Γ, and thus $\mu(\Gamma)\leq\mu(\widetilde{\Gamma})$.

This completes the proof. □

Immediately, we have the following corollary.

Corollary 6. *The metric dimension of the graph $K_{n,n}-I$, i.e. a complete bipartite graph with a 1-factor removed, is $n-1$.*

Proof. This graph is the bipartite double of the complete graph K_n, which has diameter 1, odd girth 3, and metric dimension $n-1$. □

The *Odd graph* O_k has as its vertex set the collection of all $(k-1)$-subsets of a $(2k-1)$-set, with two vertices adjacent if and only if the corresponding $(k-1)$-sets are disjoint. The Odd graph O_3 is the Petersen graph. This graph is distance-regular, has diameter $k-1$ and odd girth $2k-1$, so therefore satisfies the conditions of Theorem 3(i); its bipartite double is known as the *doubled Odd graph*. (See [9, §9.1D] for further details.) Consequently, we have another corollary to Theorem 5.

Corollary 7. *The Odd graph O_k and doubled Odd graph $\widetilde{O}_k$ have equal metric dimension, which is at most $2k-2$.*

Proof. It follows from [3, Theorem 6] that the Odd graph O_k has metric dimension at most $2k-2$. Since this graph satisfies the hypotheses of Theorem 5, the result follows. □

This latter corollary provides a slight improvement on Theorem 3.1 of Guo, Wang and Li [18], who showed that $\mu(\widetilde{O}_k) \leq 2k-1$.

3 Taylor graphs

A *Taylor graph* is 2-antipodal distance regular graph on $2n+2$ vertices, obtained via the following construction, due to Taylor and Levingston [30]. Suppose that $\Delta = (V,E)$ is a strongly regular graph with parameters $(n,2c,a,c)$. Construct a new graph Γ by taking two copies of the set V labelled as V^+, V^-, along with two new vertices ∞^+, ∞^-, and defining adjacency as follows: let ∞^+ be adjacent to all of V^+, ∞^- be adjacent to all of V^-, $u^+ \sim v^+$ and $u^- \sim v^-$ (in Γ) if and only if $u \sim v$ (in Δ), and $u^+ \sim v^-$ if and only if $u \neq v$ and $u \not\sim v$ (where $\sim$ denotes adjacency).

From the construction, one may verify that Γ is indeed distance-regular, 2-antipodal, and that the antipodal quotient is a complete graph K_{n+1}. The given labelling of the vertices ensures that v^+ is the unique antipode of v^-, for all $v \in V \cup \{\infty\}$. For any vertex x of Γ, let $\Gamma[x]$ denote the subgraph of Γ induced on the set of neighbours of x. The construction ensures that Δ is isomorphic to both $\Gamma[\infty^+]$ and $\Gamma[\infty^+]$; for any other vertex x, $\Gamma[x]$ is also strongly regular with the same parameters, but need not be isomorphic to Δ. As a simple example, one may use this construction to obtain the icosahedron from a 5-cycle.

A *two-graph* $\mathcal{D}$ is a pair $(\Omega, \mathcal{B})$, where Ω is a set and $\mathcal{B}$ is a collection of 3-subsets of Ω, with the property that any 4-subset of Ω contains an even number of members of $\mathcal{B}$. One may construct a two-graph from a graph with vertex set Ω, by taking the triples of vertices which contain an odd number of edges. Then there is an equivalence between two-graphs and equivalence class of graphs under the operation of *Seidel switching*: this operation partitions the vertex set into two parts, retaining all edges within each part, and "switching" the edges across the partition by interchanging edges and non-edges. Conversely, from any element $x \in \Omega$, one may form a graph with vertex set Ω by deleting x from all triples which contain it, and taking the resulting pairs as edges; such a graph is a *descendant* of $\mathcal{D}$. The descendants of Ω are precisely the members of the corresponding switching class which contain an isolated vertex. By abuse of terminology, we will use the term "descendant" to refer to the graph with the isolated vertex removed. For more information on two-graphs and switching classes, see [24, 28].

A two-graph is *regular* if every 2-subset of Ω occurs in a constant number of members of $\mathcal{B}$. In [29], Taylor proved that the descendants of a regular two-graph on $n+1$ points are necessarily strongly regular graphs with parameters $(n, 2c, a, c)$, and any such graphs in the same switching class give rise to the same two-graph. Taylor and Levingston [30] subsequently showed the following; see also [9, §1.5] for an account of their work.

Theorem 8 (Taylor and Levingston [30])**.**

(i) An antipodal 2*-cover of* K_{n+1} *is necessarily a Taylor graph.*

(ii) There exists a one-to-one correspondence between Taylor graphs and regular two-graphs on $n+1$ *points.*

(iii) The isomorphism classes of descendants of a regular two-graph $\mathcal{D}$*, i.e. the members of a switching class of strongly regular graphs with parameters* $(n, 2c, a, c)$*, are precisely the isomorphism classes of induced subgraphs* $\Gamma[v]$ *of the corresponding Taylor graph* Γ*.*

To confuse matters, the strongly regular graphs which are the descendants of a regular two-graph arising from the group $\mathrm{PSU}(3, q^2)$, as discovered by Taylor [29], are sometimes referred to as "Taylor's graph": see [27].

The main result of this section is to relate the resolving sets for a Taylor graph with those for the descendants of the corresponding regular two-graph.

Theorem 9. *Let* $\mathcal{D}$ *be a regular two-graph with corresponding Taylor graph* Γ*, and let* $\{\Delta_1, \ldots, \Delta_s\}$ *be the switching class of descendants of* $\mathcal{D}$*. Choose a descendant* Δ *with the smallest metric dimension, i.e.* $\mu(\Delta) \leq \mu(\Delta_i)$ *for all descendants* Δ_i*. Then we have:*

(i) $\mu(\Gamma) = \mu(\Delta) + 1$*; and*

(ii) $\mu(\Delta_i) \in \{\mu(\Delta), \mu(\Delta)+1\}$ *for all descendants* Δ_i*.*

Proof. First, we show that $\mu(\Gamma) \leq \mu(\Delta)+1$. Label the vertices of Γ as $V^+ \cup V^- \cup \{\infty^+, \infty^-\}$, as described above, and choose a smallest resolving set $R \subseteq V$ for Δ.

We will show that $R^+ \cup \{\infty^+\}$ is a resolving set for Γ. Since R is a resolving set for Δ, then for any pair of distinct vertices $u, v \in V$, there exists $x \in R$ such that $\mathrm{d}_\Delta(u,x) \neq \mathrm{d}_\Delta(v,x)$. Since $\mathrm{d}_\Delta(u,x) = \mathrm{d}_\Gamma(u^+,x^+)$ and $\mathrm{d}_\Delta(v,x) = \mathrm{d}_\Gamma(v^+,x^+)$, it follows that x^+ resolves the pair (u^+,v^+). Likewise, x^- resolves the pair (u^-,v^-); however, since Γ is 2-antipodal, Lemma 1 shows that x^+ will also resolve the pair (u^-,v^-). Any pair of vertices of the form (u^+,v^-) will be resolved by ∞^+, as $\mathrm{d}_\Gamma(u^+,\infty^+) = 1$ for any $u^+ \in V^+$, and $\mathrm{d}_\Gamma(v^-,\infty^+) = 2$ for any $v^- \in V^-$. Finally, any pair involving one of ∞^+ or ∞^- will be resolved by ∞^+, since ∞^- is the unique vertex at distance 3 from ∞^+.

Now we will establish the reverse inequality, i.e. $\mu(\Gamma) \geq \mu(\Delta)+1$. Choose a resolving set S for Γ of size $\mu(\Gamma)$. Now choose some vertex $x \in S$, and consider the subgraph $\Gamma[x]$ induced on the set $N(x)$ of neighbours of x. Since Γ is a Taylor graph, $\Gamma[x]$ must be isomorphic to a descendent Δ_i of the regular two-graph $\mathcal{D}$, and thus has diameter 2. Furthermore, the vertices in $\{x\} \cup N(x)$ form one part of a 2-antipodal partition, so by applying Lemma 1, we may assume that the remaining vertices of S are all neighbours of x.

Since S is a resolving set for Γ, then for any $u,v \in N(x)$, there exists a vertex $w \in S$ that resolves the pair (u,v); note that $w \neq x$, as x is clearly adjacent to all of its neighbours. Furthermore, for any pair of vertices $u,v \in N(x)$, we have that $\mathrm{d}_\Gamma(u,v) = \mathrm{d}_{\Gamma[x]}(u,v)$: since $\Gamma[x]$ is an induced subgraph, u and v are adjacent in Γ if and only if they are adjacent in $\Gamma[x]$, while if u and v are not adjacent, they have distance 2 in Γ (in a path through x) and distance 2 in $\Gamma[x]$ (since it has diameter 2). As we assumed that $S \setminus \{x\} \subseteq N(x)$, this shows that $S \setminus \{x\}$ is a resolving set of size $\mu(\Gamma)-1$ for $\Gamma[x]$.

Consequently, we have

$$\mu(\Delta) \leq \mu(\Delta_i) = \mu(\Gamma[x]) \leq \mu(\Gamma) - 1,$$

as required, and this concludes the proof of part (i).

To prove part (ii), we note that a given descendant Δ_i need not arise in the manner described above, i.e. induced on the set of neighbours of a vertex x of a minimum resolving set for Γ. However, any resolving set for Γ may be used to construct a resolving set of the same size for Δ_i. Suppose that $\Delta_i \cong \Gamma[w]$ for some vertex w. If S is a minimum resolving set for Γ that does not contain w, then we can still apply Lemma 1 to assume that $S \subseteq N(w)$, and the same argument as above shows that S is also a resolving set for $\Gamma[w]$. Therefore, $\mu(\Delta_i) \leq \mu(\Gamma)$, and we have

$$\mu(\Delta) \leq \mu(\Delta_i) \leq \mu(\Gamma) = \mu(\Delta) + 1,$$

and part (ii) follows. □

We remark that in the case of distance-transitive Taylor graphs (such as those obtained from Paley graphs), all descendants are isomorphic, and the result simply states $\mu(\Gamma) = \mu(\Delta) + 1$.

The result in part (ii) of Theorem 9 seems a little unsatisfactory: a better result would be that all strongly regular graphs in the same switching class have the same metric dimension, although the author was unable to show this. There is computational evidence to support such a claim. It is known that strongly regular graphs with the same parameters need not have the same metric dimension: the Paley graph on 29 vertices has metric dimension 6, while the other strongly regular graphs with parameters $(29,14,6,7)$, which fall into five switching classes, all have metric dimension 5 (see [2, Table 2]). Furthermore, the 3854 strongly regular graphs with parameters $(35,16,6,8)$, which fall into exactly 227 switching classes [23], all have metric dimension 6 (see [2, Table 13]). (As an application of Theorem 9, we know that all 227 Taylor graphs on 72 vertices have metric dimension 7.)

It is known that primitive strongly regular graphs (i.e. those which are not complete multipartite graphs) have metric dimension logarithmic in the number of vertices: see [4, §3.7] for a discussion of this. Consequently, we can combine this with Theorem 9 to obtain an asymptotic result on the metric dimension of Taylor graphs.

Corollary 10. *Suppose that Γ is a Taylor graph with $2n+2$ vertices. Then $\mu(\Gamma) = \Theta(\log n)$.*

Proof. Suppose Δ is a descendant with the smallest metric dimension. Since Δ has diameter 2, it follows that $n \leq \mu(\Delta) + 2^{\mu(\Delta)}$ (cf. [4, Proposition 3.6]), and thus $\mu(\Delta) > \log_2 n - 1$. Also, a result of Babai [1] (see also [4, Theorem 3.31]) implies that, since Δ is strongly regular, $\mu(\Delta) < 8 \log n$. Since $\mu(\Gamma) = \mu(\Delta) + 1$, the result follows. □

In particular, in the case of Paley graphs, the constant in the upper bound can be improved from 8 to 2 (see Fijavž and Mohar [15]), which gives a tighter bound on the metric dimension of the corresponding Taylor graphs.

4 Incidence graphs of symmetric designs

A *symmetric design* (or *square* 2*-design*) with parameters (v,k,λ) is a pair $(X,\mathcal{B})$, where X is a set of v *points*, and $\mathcal{B}$ is a family of k-subsets of X, called *blocks*, such that any pair of distinct points are contained in exactly λ blocks, and that any pair of distinct blocks intersect in exactly λ points. It follows that $|\mathcal{B}| = v$. The *incidence graph* of a symmetric design is the bipartite graph with vertex set $X \cup \mathcal{B}$, with a point $x \in X$ adjacent to block $B \in \mathcal{B}$ if and only if $x \in B$. It is straightforward to show that the incidence graph of a symmetric design is a bipartite distance-regular graph with diameter 3. The converse is also true (see [9, §1.6]): any bipartite distance-regular graph of diameter 3 gives rise to a symmetric design.

The *dual* of a symmetric design is the design obtained from the incidence graph by reversing the roles of points and blocks; $(X,\mathcal{B})$ and its dual both have the same parameters. The *complement* of a symmetric design $(X,\mathcal{B})$ has the same point set X, and block set $\overline{\mathcal{B}} = \{X \setminus B : B \in \mathcal{B}\}$. The incidence graph of $(X,\overline{\mathcal{B}})$ is obtained from that of $(X,\mathcal{B})$ by interchanging edges and non-edges across the bipartition (i.e. taking the "bipartite complement"). If $(X,\mathcal{B})$ has parameters (v,k,λ), then $(X,\overline{\mathcal{B}})$ has parameters $(v,v-k,v-2k+\lambda)$.

Suppose Γ is the incidence graph of $(X,\mathcal{B})$. We note that distances in Γ are as follows:

$$\begin{aligned} \mathrm{d}_\Gamma(x,B) &= \begin{cases} 1 & \text{if } x \in B, \\ 3 & \text{if } x \notin B, \end{cases} \\ \mathrm{d}_\Gamma(x,y) &= 2, \\ \mathrm{d}_\Gamma(A,B) &= 2, \end{aligned}$$

for any distinct points $x,y \in X$ and any distinct blocks $A,B \in \mathcal{B}$. It follows that the incidence graph of a symmetric design and that of ts complement have the same metric dimension; clearly, this holds for the incidence graph of a symmetric design and its dual, as the incidence graphs are isomorphic.

The distance-2 graph of Γ is the disjoint union of two complete graphs K_v; consequently, if a resolving set R for Γ is contained entirely within X or entirely within $\mathcal{B}$, we have $|R| \geq v-1$. In the case of the trivial symmetric design with $k = v-1$, where the blocks are all the $(v-1)$-subsets, the graph obtained is $K_{v,v} - I$, which has metric dimension $v-1$ by Corollary 6. However, for non-trivial symmetric designs, it is natural to ask if smaller resolving sets exist, which must therefore contain both types of vertex. A natural way to construct a resolving set is as follows.

Suppose Γ is the incidence graph of a symmetric design $(X,\mathcal{B})$. A *split resolving set* for Γ is a set $R = R_X \cup R_{\mathcal{B}}$, where $R_X \subseteq X$ and $R_{\mathcal{B}} \subseteq \mathcal{B}$, chosen so that any two points x,y are resolved by a vertex in $R_{\mathcal{B}}$, and any two blocks A,B are resolved by a vertex in R_X. We call R_X and $R_{\mathcal{B}}$ *semi-resolving sets*. The smallest size of a split resolving set will be denoted by $\mu^*(\Gamma)$. We note that a split resolving set is itself a resolving set, as any vertex will resolve a pair x,B, given that the parities of the distances to x and to B will be different; therefore, we only need consider resolving point/block pairs. Clearly, we have $\mu(\Gamma) \leq \mu^*(\Gamma)$.

A straightforward observation is that, for any two blocks A,B, the point x resolves the blocks A,B if and only if x lies in exactly one of the two blocks (i.e. $x \in A$ and $x \notin B$, or vice-versa), and a block B resolves the points x,y if and only if exactly one of x,y lies in B.

4.1 Projective planes

A symmetric design with $\lambda = 1$ is a *projective plane*. In this case, the blocks of the design are usually referred to as *lines*, and are denoted by $\mathcal{L}$. It is known that, for a projective plane to exist, we have $v = q^2 + q + 1$ and $k = q + 1$ for some integer q, called the *order* of the projective plane.

A *blocking set* for a projective plane $\Pi = (P, \mathcal{L})$ of order q is a subset of points $S \subseteq P$ chosen so that every line $L \in \mathcal{L}$ contains at least one point in S; moreover, S is a *double blocking set* if every line L contains at least two points in S. Ball and Blokhuis [6] showed that, for $q > 3$, a double blocking set has size at least $2(q + \sqrt{q} + 1)$, with equality occurring in the plane $\mathrm{PG}(2,q)$ when q is a square. Also, one can easily construct a double blocking set of size $3q$ by taking the points of three non-concurrent lines. Double blocking sets and semi-resolving sets are related by the following straightforward proposition.

Proposition 11. *A double blocking set with a single point removed forms a semi-resolving set for the lines of a projective plane.*

Proof. Let S be a double blocking set for $\Pi = (P, \mathcal{L})$. Any pair of distinct lines L_1, L_2 intersects in a unique point x. Since S is a double blocking set, there exists $y \in L_1 \setminus \{x\}$ such that $y \in S$ and $y \notin L_2$. Hence y resolves the lines L_1, L_2. By the same argument, there also exists $z \in L_2 \setminus \{x\}$ such that $z \in S$ and $z \notin L_1$. This redundancy allows us to delete a point from S and still have a semi-resolving set; however, deleting two points from S may prevent us from resolving some pairs of lines. □

By taking a semi-resolving set of this form for the points, along with the dual of such a set, we obtain a split resolving set for Γ_Π of size $(\tau_2(\Pi) - 1) + (\tau_2(\Pi^\perp) - 1)$ (where $\tau_2(\Pi)$ denotes the smallest size of a double blocking set in Π, and $\Pi^\perp$ denotes the dual plane); if Π is self-dual then this simplifies as $2(\tau_2(\Pi) - 1)$. At the problem session of the 2011 British Combinatorial Conference, the author asked whether this was best possible. In 2012, the question was answered by Héger and Takáts [21] for the Desarguesian plane $\mathrm{PG}(2,q)$.

Theorem 12 (Héger and Takáts [21, Theorem 4]). *A semi-resolving set for* $\mathrm{PG}(2,q)$ *has size at least* $\min\{2q + q/4 - 3, \tau_2(\mathrm{PG}(2,q)) - 2\}$*; for a square prime power* $q \geq 121$*, this is at least* $q + \sqrt{q}$.

Of course, a minimum resolving set for Γ_Π need not be a split resolving set. W. J. Martin (personal comunication) was able to construct a non-split resolving set for Γ_Π of size $4q - 4$ (see [21, Figure 1]), and conjectured that this was best possible (except for small orders). This conjecture was also proved in the 2012 paper of Héger and Takáts [21].

Theorem 13 (Héger and Takáts [21, Theorem 2]). *For a projective plane* Π *of order* $q \geq 23$*, the metric dimension of its incidence graph* Γ_Π *is* $\mu(\Gamma_\Pi) = 4q - 4$.

Héger and Takáts also gave a complete description of all resolving sets of this size: see [21, §3].

4.2 Biplanes

Symmetric designs with $\lambda = 2$ are known as *biplanes* [11]. For a biplane to exist, we must have $v = \binom{k}{2} + 1$. Unlike the case of projective planes, there are no known infinite families of biplanes. In fact only 16 examples are known (see [22]), the largest having $v = 79$ points and lines of size $k = 13$ (we continue with the geometric terminolgy here). Nevertheless, a split resolving set for a biplane is quite straightforward to construct.

Proposition 14. *Let Γ be the incidence graph of a $(v,k,2)$ biplane $(X,\mathcal{L})$ with $k \geq 4$. Then any $k-1$ collinear points form a semi-resolving set for the lines of $(X,\mathcal{L})$, and thus $\mu(\Gamma) \leq \mu^*(\Gamma) \leq 2k-2$.*

Proof. Choose a distinguished line L and any $(k-1)$-subset $R_X \subset L$. We will show that for any two lines $L_1, L_2 \in \mathcal{L}$ that there exists a point in R_X which resolves L_1, L_2, i.e. which lies on exactly one of the two lines.

First, if $L = L_1$, then L contains only two points of L_2, and so there exist points in $R_X \subset L$ not on L_2; the case $L = L_2$ works similarly. So we suppose L, L_1 and L_2 are all distinct. There can exist at most one point lying on all three lines, so we consider the two cases separately. First, suppose $L \cap L_1 = \{x,y\}$ and $L \cap L_2 = \{a,b\}$ (where a,b,x,y are all distinct). Any of these four points lies on exactly one of the lines L_1, L_2, and thus resolves this pair of lines; at most one of these points is not in R_X. Second, if $L \cap L_1 = \{x,y\}$ and $L \cap L_2 = \{x,z\}$ (where x,y,z are all distinct), then L_1, L_2 can be resolved by either y or z, and again at most one of these is not in R_X.

By taking this semi-resolving set, along with an equivalent semi-resolving set in the dual design (i.e. a collection of $k-1$ concurrent lines), we obtain a split resolving set of size $2k-2$. □

This result gives an upper bound on $\mu(\Gamma)$ of $O(\sqrt{v})$. However, in certain situations, we can actually improve this to $\Theta(\log v)$, as the next section demonstrates.

4.3 Symmetric designs with a null polarity

A *polarity* of a symmetric design $(X,\mathcal{B})$ is a bijection $\sigma : X \to \mathcal{B}$ which preserves the point/block incidence relation. It is straightforward to see that $(X,\mathcal{B})$ admits a polarity if and only if there is an ordering of the points and blocks so that the incidence matrix of the design is symmetric. A point is called *absolute* if it is incident to its image under σ. A polarity σ is said to be *null* if no points are absolute.[1] In this situation, the incidence matrix has zero diagonal, and so is the adjacency matrix of a graph Δ; this graph is strongly regular with parameters (v,k,λ,λ). We observe that a symmetric design $(X,\mathcal{B})$ may admit more than one null polarity, and the corresponding strongly regular graphs need not be isomorphic. (See Cameron and van Lint [12] for more details on these topics.)

Conversely, if one has a strongly regular graph $\Delta = (V,E)$ with parameters (v,k,λ,λ), the bipartite double of that graph (recall Section 2) is the incidence graph of a symmetric design with parameters (v,k,λ), which admits a null polarity in an obvious way: the points

[1] Sometimes, the term "null polarity" is used when *all* points are absolute; however, this is equivalent to the complement of the design having no absolute points.

and blocks may be labelled by V^+ and V^- respectively, and the map $\sigma : v^+ \mapsto v^-$ is a null polarity. We note that non-isomorphic graphs may give rise to the same symmetric design: for instance, the 4×4 lattice $H(2,4)$ and the Shrikhande graph are non-isomorphic strongly regular graphs with parameters $(16,6,2,2)$, yet their bipartite doubles are isomorphic (and give rise to a biplane).

Given this relationship with bipartite doubles, one may ask if there is a result similar to Theorem 5 which can be applied here to find the metric dimension of Γ, and we have the following theorem.

Theorem 15. *Let Γ be the incidence graph of a non-trivial (v,k,λ) symmetric design with a null polarity, and let Δ be a corresponding strongly regular graph with parameters (v,k,λ,λ). Then $\mu(\Gamma) \leq 2\mu(\Delta)$.*

Proof. Since Γ is the bipartite double of Δ, we will label the points and blocks of the design by V^+ and V^- respectively. Suppose $R \subseteq V$ is a resolving set for Δ; we will show that $R^+ \cup R^-$ is a resolving set for Γ. Now, distances in Γ are as follows:

$$\begin{aligned} \mathrm{d}_\Gamma(u^+,w^-) &= \begin{cases} 1 & \text{if } u \sim w \text{ in } \Delta, \\ 3 & \text{if } u \not\sim w \text{ in } \Delta, \end{cases} \\ \mathrm{d}_\Gamma(u^+,w^+) &= 2, \\ \mathrm{d}_\Gamma(u^-,w^-) &= 2, \end{aligned}$$

where $u \neq w$. Clearly, any vertex resolves u^+, w^- (as the distances will have different parities), so it suffices to consider resolving pairs of vertices of the form u^+, w^+ and u^-, w^-.

If $u \in R$, then clearly u^+ resolves the pair u^+, w^+ and u^- resolves the pair u^-, w^- (and likewise if $w \in R$), so we assume that $u, w \notin R$. Since R is a resolving set for Δ, there exists $x \in R$ where $\mathrm{d}_\Delta(u,x) \neq \mathrm{d}_\Delta(w,x)$; without loss of generality, this implies that $u \sim x$ and $w \not\sim x$, so therefore $\mathrm{d}_\Gamma(u^+,x^-) = 1$ and $\mathrm{d}_\Gamma(w^+,x^-) = 3$, and thus x^- resolves the pair u^+, w^+. Similarly, x^+ resolves the pair u^-, w^-. Hence any pair of vertices of Γ is resolved by a vertex in $R^+ \cup R^-$, and we are done. □

Immediately, we have the following corollary, which is reminiscent of Corollary 10 for Taylor graphs.

Corollary 16. *Let Γ be the incidence graph of a non-trivial (v,k,λ) symmetric design with a null polarity. Then $\mu(\Gamma) = \Theta(\log v)$.*

Proof. Since Γ has $2v$ vertices and diameter 3, we have $2v \leq \mu(\Gamma) + 3^{\mu(\Gamma)}$, and thus $\mu(\Gamma) > \log_3(2v) - 1$. By the result of Babai [1], the strongly regular graph Δ satisfies $\mu(\Delta) < 8 \log v$. Since $\mu(\Gamma) \leq 2\mu(\Delta)$, the result follows. □

We remark that this result can never be applied to the incidence graphs of projective planes, since it is known that any polarity of a projective plane of order q has between $q+1$ and $q\sqrt{q}+1$ absolute points (see [10, Proposition 4.10.1]), and thus a projective plane cannot admit a null polarity. Therefore, the fact that the metric dimension of the incidence graph of a projective plane has size $\Theta(\sqrt{v})$ (from Theorem 13) is not contradicted here. However, there do exist biplanes which admit a null polarity, such as the biplane with parameters $(16,6,2)$ referred to above, or a $(56,11,2)$ biplane arising from the Gewirtz graph. (Whether there exist infinite families of biplanes, for which an asymptotic result would make sense, is unknown.)

5 Conclusion

The underlying theme behind all of the main results of this paper is one of reducing imprimitive graphs to primitive ones: we were able to obtain results about the metric dimension of various classes of imprimitive distance-regular graphs by indicating a connection to the metric dimension of related primitive distance-regular (or strongly regular) graphs. It seems plausible that this is part of a deeper theory: the results in this paper are intended as the beginnings of such a study. It may be that some of the results can be strengthened to weaker hypotheses: for instance, one could ask if $\mu(\Gamma) = \Theta(\log v)$ for the incidence graphs of arbitrary (v,k,λ) symmetric designs with $\lambda > 1$, without the assumption of a null polarity.

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