On the Classification of Binary Completely Transitive Codes with Almost-Simple Top-Group

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Abstract

A code C in the Hamming metric, that is, is a subset of the vertex set $V\Gamma$ of the Hamming graph $\Gamma = H(m, q)$, gives rise to a natural *distance partition* $\{C, C_1, \ldots, C_\rho\}$, where ρ is the covering radius of C. Such a code C is called *completely transitive* if the automorphism group $\mathrm{Aut}(C)$ acts transitively on each of the sets C, C_1, ..., C_ρ. A code C is called 2-neighbour-transitive if $\rho \geqslant 2$ and $\mathrm{Aut}(C)$ acts transitively on each of C, C_1 and C_2.

Let C be a completely transitive code in a binary ($q = 2$) Hamming graph having full automorphism group $\mathrm{Aut}(C)$ and minimum distance $\delta \geqslant 5$. Then it is known that $\mathrm{Aut}(C)$ induces a 2-homogeneous action on the coordinates of the vertices of the Hamming graph. The main result of this paper classifies those C for which this induced 2-homogeneous action is *not* an affine, linear or symplectic group. We find that there are 13 such codes, 4 of which are non-linear codes. Though most of the codes are well-known, we obtain several new results. First, a new non-linear completely transitive code is constructed, as well as a related non-linear code that is 2-neighbour-transitive but not completely transitive. Moreover, new proofs of the complete transitivity of several codes are given. Additionally, we answer the question of the existence of distance-regular graphs related to the completely transitive codes appearing in our main result.

1   Introduction

A subset C of the vertex set $V\Gamma$ of the Hamming graph $\Gamma = H(m, q)$ is called a *code*, the elements of C are called *codewords*, and the subset C_i of $V\Gamma$ consisting of all vertices of $H(m, q)$ having nearest codeword at Hamming distance i is called the *set of i-neighbours* of C. Some important generalisations of perfect codes were introduced in the 1970s: *uniformly packed codes*, introduced by Semakov, Zinoviev and Zaitsev [35], and the classes of *completely regular* and

s-regular codes, defined by Delsarte [12]. These generalisations involve certain combinatorial regularity conditions on the *distance partition* $\{C, C_1, \ldots, C_\rho\}$ of a code C, where ρ is the *covering radius*.

The parameters of perfect codes over prime-power alphabets have been classified, and codes satisfying these parameters found; see [38] or [41]. In contrast, the class of completely regular codes is vast, with similar classification results remaining an active area of research. Several recent results have been obtained by Borges et al. [4, 6, 7]. For a survey of results on completely regular codes see [8].

The focus of current paper is the algebraic analogues, defined directly below, of the classes of completely regular and s-regular codes. Note that the group $\mathrm{Aut}(C)$ is the setwise stabiliser of C in the full automorphism group of $H(m, q)$. In particular, if C is linear then we consider the the group of translations by codewords of C to be contained in $\mathrm{Aut}(C)$.

Definition 1.1. Let C be a code in $H(m, q)$ with covering radius ρ, let $s \in \{1, \ldots, \rho\}$, and $X \leqslant \mathrm{Aut}(C)$. Then C is said to be:

1. (X, s)-*neighbour-transitive* if X acts transitively on each of the sets $C, C_1, \ldots, C_s$.

2. X-*neighbour-transitive* if C is $(X, 1)$-neighbour-transitive.

3. X-*completely transitive* if C is (X, ρ)-neighbour-transitive.

4. s-*neighbour-transitive* if C is $(\mathrm{Aut}(C), s)$-neighbour-transitive, *neighbour-transitive* if C is $\mathrm{Aut}(C)$-neighbour-transitive, or *completely transitive* if C is $\mathrm{Aut}(C)$-*completely transitive*.

A variant of the above concept of complete transitivity was introduced for linear codes by Solé [36], with the above definition first appearing in [23]. Completely transitive codes form a subfamily of completely regular codes, and s-neighbour transitive codes are a sub-family of s-regular codes, for each s. Characterisation of certain families of neighbour-transitive codes in Hamming graphs have been achieved (see [19, 20, 21]), as well as several classification results of subfamilies of neighbour-transitive codes in Johnson graphs (see [14, 28, 31, 34]). The results in this paper depend upon the series of papers [16, 17, 18, 26] on 2-neighbour-transitive codes, which form part of the second author's PhD thesis.

Each vertex of $H(m, q)$ is of the form $\alpha = (\alpha_1, \ldots, \alpha_m)$, where the entries α_i come from an *alphabet* Q of size q. A typical automorphism of $H(m, q)$ is a composition of two automorphisms, each of a special type. An automorphism of the first type corresponds to an m-tuple $(h_1, \ldots, h_m)$ of permutations of Q and maps a vertex α to $(\alpha_1^{h_1}, \ldots, \alpha_m^{h_m})$. An automorphism of the second type corresponds to a permutation of the set $M = \{1, \ldots, m\}$ of subscripts and simply permutes the entries of vertices, for example the map corresponding to the permutation (123) of M maps $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots, \alpha_m)$ to $(\alpha_3, \alpha_1, \alpha_2, \alpha_4, \ldots, \alpha_m)$. (More details are given in Section 2.1.)

Our main result is Theorem 1.3, further below, the proof of which relies on [26, Theorem 1.2], stated below as Theorem 1.5. For now we give a brief discussion of the relation between the two theorems, only what is required to properly state Theorem 1.3. If a completely transitive code C has minimum distance at least 5 then, since this implies C_2 is non-empty, we have that C is by definition 2-neighbour-transitive. Hence, Theorem 1.5 applies. Recall that the *socle* $\mathrm{soc}(G)$ of a group G is the product of all its minimal normal subgroups. In particular, Theorem 1.5 gives all possibilities for the socle of the action on M of the stabiliser $\mathrm{Aut}(C)_{\mathbf{0}}$ of the codeword $\mathbf{0}$ (assumed to be in C) for any 2-neighbour-transitive code C with minimum distance at least 5. If $|C| > 2$ then this socle is either one of the Mathieu groups M_{11} or M_{12}, as in Theorem 1.5 part

G	m	conditions
$\mathrm{PSL}_3(4)$	21	-
A_7	15	-
$\mathrm{PSL}_2(r)$	$r+1$	$23 \leqslant r \equiv \pm 1 \pmod 8$
$\mathrm{PSU}_3(r)$	r^3+1	r is odd
$\mathrm{Ree}(r)$	r^3+1	$r \geqslant 3$
M_m	$11, 12, 22, 23, 24$	-
HS	176	-
Co_3	276	-

Table 1: Groups G for which there exists a non-trivial binary 2-neighbour-transitive code C in $H(m,2)$ with $G \cong \mathrm{soc}(\mathrm{Aut}(C)_{\mathbf{0}}^M)$ and G is not isomorphic to i) $\mathrm{PSL}_t(2^k)$ unless $(t,k) = (3,2)$, ii) $\mathrm{Sp}_{2t}(2)$ for $t \geqslant 3$, or iii) $\mathbb{Z}_2^t$ for $t \geqslant 3$.

2, or contained in [26, Table 1], as in Theorem 1.5 part 3. Table 1 lists the groups we consider here (see the following remark) for the socle of the action of $\mathrm{Aut}(C)_{\mathbf{0}}$ on M.

Remark 1.2. The groups appearing in [26, Table 1] but not in Table 1, and hence not covered in this paper are: $\mathrm{PSL}_t(2^k)$ with $(t,k) \neq (3,2)$; $\mathrm{Sp}_{2t}(2)$ for $t \geqslant 3$; and $\mathbb{Z}_2^t$ for $t \geqslant 3$. This is due to the fact that the first two cases require more technical representation theory than those treated in this paper, while the third case requires the classification of finite transitive linear groups to be applied. However, Table 1 does contain $\mathrm{PSL}_3(4)$. In this case the corresponding codes are related to the geometry of $\mathrm{PG}_2(4)$, which also arises in the treatment of the larger Mathieu groups; see Examples 3.3 and 3.4, and [13, Sections 6.5–6.7].

We can now state Theorem 1.3. Note: we say that a code C is *non-trivial* if $|C| > 2$; if C is a linear code of length m, dimension k, minimum distance δ and covering radius ρ then we say it has *parameters* $[m, k, \delta; \rho]$; if C is a non-linear code of length m, minimum distance δ and covering radius ρ then we say it has *parameters* $(m, |C|, \delta; \rho)$.

Theorem 1.3. *Let C be a non-trivial binary completely transitive code in $H(m,2)$ with minimum distance $\delta \geqslant 5$ and suppose the socle of the action of $\mathrm{Aut}(C)_{\mathbf{0}}$ on M is isomorphic to G as in one of the lines of Table 1. Then C is equivalent to one of the codes given in Table 2. Conversely, each code in Table 2 is completely transitive.*

While this paper is concerned with classifying certain subfamilies of binary completely transitive codes with minimum distance at least 5, many completely transitive codes having minimum distance less then 5 are known. For instance, [7] exhibits completely transitive codes with minimum distances 3 and 4, whilst [5] constructs completely transitive codes with minimum distances 3 and below. For more examples of such codes, see [8]. It is worth mentioning that Borges et al. [3] proved that any binary linear completely transitive code has error correction capacity $e = \lfloor (\delta - 1)/2 \rfloor \leqslant 3$, and all codes arising in Theorem 1.3 also satisfy $e \leqslant 3$.

As part of our classification we also construct two non-linear codes that, to the best of our knowledge, have not previously appeared in the literature. These are the 2-neighbour-transitive code $\langle \mathcal{P}, \Delta_2 \rangle \cup \langle \mathcal{P}, \Delta_3 \rangle$ with parameters $(21, 2^9 \cdot 3, 6; 6)$ (see Example 3.3 and Lemma 6.6) and the completely transitive code $\langle \mathcal{L}, \Delta_2 \rangle \cup \langle \mathcal{L}, \Delta_3 \rangle$ with parameters $(21, 2^{10} \cdot 3, 5; 6)$ (see Example 3.3 and Lemma 6.7). Except for $\langle \mathcal{L}, \Delta_2 \rangle \cup \langle \mathcal{L}, \Delta_3 \rangle$, all of the codes appearing in Table 2 are known to be completely regular (see, for instance, [8]). While the majority of codes in Table 2 are also known to be completely transitive, it appears that our proofs that $\langle \mathcal{L}, \Delta_1 \rangle$, $\mathcal{L}$, $\mathcal{G}_{23}^{\perp}$, $\mathcal{E}_{22}$ and $\mathcal{S}_{22}$ are completely transitive are new (see Section 3 and Lemmas 5.2, 5.4 and 6.5).

line	C	$\mathrm{Aut}(C)$	parameters	dist. reg. graph
1	$\mathcal{H}$	$2\,\mathrm{M}_{12}$	$(12, 24, 6; 3)$	no
2	$\mathcal{PH}$	$2 \rtimes \mathrm{M}_{11}$	$(11, 24, 5; 3)$	no
3	$\mathcal{NR}$	$2^5 \rtimes \mathrm{A}_8$	$(15, 256, 5; 3)$	see Section 3.1
4	$\mathcal{P}^{\perp}$	$T_C \rtimes \mathrm{P\Gamma L}_3(4)$	$[21, 12, 5; 3]$	[9, Theorem 11.3.6]
5	$\langle \mathcal{L}, \Delta_1 \rangle$	$T_C \rtimes \mathrm{P\Sigma L}_3(4)$	$[21, 11, 5; 6]$	[9, Section 11.3.H b)]
6	$\langle \mathcal{L}, \Delta_1 \rangle \cup \langle \mathcal{L}, \Delta_2 \rangle$	$T_{\mathcal{L}} \rtimes \mathrm{P\Gamma L}_3(4)$	$(21, 2^{10} \cdot 3, 5; 6)$	no
7	$\mathcal{L}$	$T_C \rtimes \mathrm{P\Gamma L}_3(4)$	$[21, 10, 5; 6]$	[9, Section 11.3.H c)]
8	$\mathcal{G}_{24}$	$T_C \rtimes \mathrm{M}_{24}$	$[24, 12, 8; 4]$	[9, Theorem 11.3.2]
9	$\mathcal{G}_{23}$	$T_C \rtimes \mathrm{M}_{23}$	$[23, 12, 7; 3]$	[9, Theorem 11.3.4]
10	$\mathcal{G}_{23}^{\perp}$	$T_C \rtimes \mathrm{M}_{23}$	$[23, 11, 8; 7]$	[9, Section 11.3.E]
11	$\mathcal{G}_{22}$	$T_C \rtimes (\mathrm{M}_{22} : 2)$	$[22, 12, 6; 3]$	[9, Theorem 11.3.5]
12	$\mathcal{E}_{22}$	$T_C \rtimes (\mathrm{M}_{22} : 2)$	$[22, 11, 6; 7]$	[9, Section 11.3.F]
13	$\mathcal{S}_{22}$	$T_C \rtimes \mathrm{M}_{22}$	$[22, 11, 7; 6]$	[9, Section 11.3.H a)]

Table 2: Non-trivial binary completely transitive codes C with minimum distance at least 5 and where $\mathrm{Aut}(C)_{\mathbf{0}}^M$ has as socle one of the groups G contained in Table 1. See Section 3 for the definitions of these codes. The last column gives a reference for a distance-regular graph related to C, or contains 'no' if no such graph can be constructed in the usual manner; see Section 3.1 for more details.

The next definition associates with any code a certain linear subcode. Note that the condition that $T_{C_{\max}} \leqslant \mathrm{Aut}(C)$ is included to ensure $C_{\max}$ is uniquely defined.

Definition 1.4. Let $Q = \mathbb{F}_2$ and C be a code in $H(m, 2)$. The *maximal linear subcode* of C, denoted $C_{\max}$, is the largest linear subcode of C with the additional property that $T_{C_{\max}} \leqslant \mathrm{Aut}(C)$, where $T_{C_{\max}}$ is the group of translations by elements of $C_{\max}$.

Below we restate [26, Theorem 1.2], which is the starting point for the proof of Theorem 1.3. Note that Theorem 1.5 is presented in terms of $C_{\max}$, as in Definition 1.4, and so differs slightly from the original statement in [26]. Including the conditions on $C_{\max}$ here is intended to increase the transparency of our proof strategy. Indeed, the proof of [26, Theorem 1.2] is effectively divided up according to whether $C_{\max}$ has dimension 0, 1, or at least 2, though the notation $C_{\max}$ is not used there. Thus, we hope the reader finds it fairly straightforward to verify that the formulation below is indeed equivalent. Note also that C non-trivial implies that $|C| > 2$, and hence the binary repetition code does not appear as a possibility for C. Recall that the groups from [26, Table 1] that are relevant to Theorem 1.3 also appear here in Table 1.

Theorem 1.5. *Let C be a non-trivial binary code in $H(m, 2)$ with minimum distance at least 5 containing the codeword $\mathbf{0}$. Then C is 2-neighbour-transitive if and only if one of the following holds:*

1. *$C_{max} = \{\mathbf{0}\}$ and C is the even weight subcode $\mathcal{E}$ of the punctured Hadamard code with $m = 11$ and minimum distance 6;*

2. *C_{max} is the binary repetition code and C is one of the following codes:*

   (a) *the Hadamard code $\mathcal{H}$ with $m = 12$ and minimum distance 6,*

   (b) *the punctured Hadamard code $\mathcal{PH}$ with $m = 11$ and minimum distance 5; or,*

3. *C_{max} has dimension at least 2 and minimum distance $\delta_{max} \geqslant 5$, and there exists a subgroup $X_{\mathbf{0}} \leqslant \mathrm{Aut}(C_{max})_{\mathbf{0}}$, where $X_{\mathbf{0}}$ and m are as in [26, Table 1], such that*

(a) C_{max} is $(X, 2)$-*neighbour-transitive, where* $X = T_{C_{max}} \rtimes X_0 \leqslant \mathrm{Aut}(C)$ *with* $T_{C_{max}}$
   denoting the group of translations by elements of C_{max}*, and,*

(b) C *is the union of a set* $\mathcal{S}$ *of cosets of* C_{max}*, and* $\mathrm{Aut}(C)$ *acts transitively on* $\mathcal{S}$*.*

Hypothesis 1, stated below, will be assumed to hold in many of the following sections. Note that the condition on the dimension of C_{max} ensures that C_{max} is neither the repetition code nor its dual.

Hypothesis 1. Let C be a completely transitive code in $H(m, 2)$ with minimum distance $\delta \geqslant 5$ and let $X = \mathrm{Aut}(C)$. Let C_{max} be the maximal linear subcode of C with minimum distance δ_{max} and let X_{max} be the setwise stabiliser in X of C_{max}. Furthermore, assume that $2 \leqslant \dim(C_{max}) \leqslant m - 2$.

We also define Hypothesis 2, an extension of Hypothesis 1, which includes an additional assumption relating to the socles of the actions of X and X_{max} on M. We briefly outline the significance of this hypothesis. In Section 2 we show that $X^M \cong X/K$ and $X_{max}^M \cong X_{max}/K$, where $K = T_{C_{max}}$, and that both of these actions, of X on M and of X_{max} on M, are 2-homogeneous. Since 2-homogeneous groups are classified according to their socles, and we deal with the case that these two actions have different socles in Section 4, Hypothesis 2 will be assumed in Sections 5 to 10.

Hypothesis 2. Assume Hypothesis 1 holds, that the kernel K of the action of X on M is $T_{C_{max}}$ and that
$$\mathrm{soc}(X/K) = \mathrm{soc}(X_{max}/K).$$

In the next section, we introduce the notation used throughout the paper and various preliminary results. In particular, in Section 2.3 we prove some new results regarding completely transitive codes useful in later sections. Section 3 contains the definitions of each of the codes appearing in Table 2 or otherwise, and Section 3.1 answers the question of the existence of a distance-regular graph related to each code. Each of Sections 4 to 10 deals with some collection of groups from Table 1. Finally, Section 11 proves Theorem 1.3.

2 Preliminaries

Let the *set of coordinate entries* M and the *alphabet* Q be sets of sizes m and q, respectively, where both m and q are integers and at least 2. The vertex set $V\Gamma$ of the Hamming graph $\Gamma = H(m, q)$ consists of all strings in the alphabet Q with positions indexed by the set M, which are usually written as m-tuples. Let $Q_i \cong Q$ be the copy of the alphabet in the entry $i \in M$ so that the vertex set of $H(m, q)$ is identified with the Cartesian product
$$V\Gamma = \prod_{i \in M} Q_i.$$

An edge in Γ exists between two vertices if and only if they differ as m-tuples in exactly one entry. If α is a vertex of $H(m, q)$ and $i \in M$ then α_i refers to the value of α in the i-th entry, that is, $\alpha_i \in Q_i$, so that $\alpha = (\alpha_1, \ldots, \alpha_m)$ when $M = \{1, \ldots, m\}$. For more in depth background material on coding theory see [10] or [32].

When considering binary codes we will be flexible with our interpretation of the vertices of $H(m, 2)$. When $q = 2$ we can identify Q with $\mathbb{F}_2$ and $V\Gamma$ with the vector space $\mathbb{F}_2^m$. Let $\{e_i \mid i \in M\}$ be the standard basis for $V\Gamma \cong \mathbb{F}_2^m$. Then a vertex α of $H(m, 2)$ will be

Notation	Explanation
$\mathbf{0}$	vertex with 0 in each entry
$\mathrm{wt}(\alpha) = \lvert\operatorname{supp}(\alpha)\rvert$	weight of α
$d(\alpha, \beta) = \lvert\operatorname{diff}(\alpha, \beta)\rvert$	Hamming distance
$\Gamma_s(\alpha) = \{\beta \in V\Gamma \mid d(\alpha, \beta) = s\}$	set of s-neighbours of α
$\delta = \min\{d(\alpha, \beta) \mid \alpha, \beta \in C, \alpha \neq \beta\}$	minimum distance of C
$d(\alpha, C) = \min\{d(\alpha, \beta) \mid \beta \in C\}$	distance from α to C
$\rho = \max\{d(\alpha, C) \mid \alpha \in V\Gamma\}$	covering radius of C
$C_s = \{\alpha \in V\Gamma \mid d(\alpha, C) = s\}$	set of s-neighbours of C
$\{C = C_0, C_1, \ldots, C_\rho\}$	distance partition of C

Table 3: Hamming graph notation.

simultaneously considered to be a vector $\alpha = \sum_{i \in M} \alpha_i e_i$, a function $\alpha : M \to \mathbb{F}_2$ such that $i \mapsto \alpha_i$, and a subset $\{i \in M \mid \alpha_i = 1\}$. Note that the Hamming metric ensures that there is only one sensible way to do this. As such, if α, β are vertices of $H(m, 2)$, then so are the sum $\alpha + \beta$, the product $\alpha\beta$, the intersection $\alpha \cap \beta$, the union $\alpha \cup \beta$ and $M \setminus \alpha$, where the sum and product may be interpreted in terms of functions and the remaining operations in terms of sets. Note also that we write the binary repetition code, that is, the subspace of $\mathbb{F}_2^m$ generated by the vector $\mathbf{1} = (1, \ldots, 1)$, as $\langle\mathbf{1}\rangle$ and its dual as $\langle\mathbf{1}\rangle^\perp$.

Let α, β be vertices and C be a code in a Hamming graph $H(m, q)$ with $0 \in Q$ a distinguished element of the alphabet. A summary of important notation regarding codes in Hamming graphs is contained in Table 3.

Note that if the minimum distance δ of a code C satisfies $\delta \geqslant 2s$, then the set of s-neighbours C_s satisfies $C_s = \cup_{\alpha \in C}\Gamma_s(\alpha)$ and if $\delta \geqslant 2s + 1$ this is a disjoint union. This fact is crucial in many of the proofs below; it is often assumed that $\delta \geqslant 5$, in which case every element of C_2 is distance 2 from a unique codeword.

A *linear* code is a code C in $H(m, q)$ with alphabet $Q = \mathbb{F}_q$ a finite field, so that the vertices of $H(m, q)$ form a vector space V, such that C is an $\mathbb{F}_q$-subspace of V. Given $\alpha, \beta \in V$, the usual inner product is given by $\langle\alpha, \beta\rangle = \sum_{i \in M} \alpha_i \beta_i$. The *dual* code of C is $C^\perp = \{\beta \in V \mid \forall \alpha \in C, \langle\alpha, \beta\rangle = 0\}$.

Let C be a code in $H(m, q)$. The *punctured code*, also sometimes called the *truncated code*, of C at the coordinate $i \in M$ is the code in $H(m - 1, q)$ consisting of set of all vertices α' of $H(m-1, q)$ such that there exists an $\alpha \in C$ with $\alpha'_j = \alpha_j$ for all $j \in M\setminus\{i\}$. The *shortened code* of C at $i \in M$ is the code in $H(m - 1, q)$ consisting of set of all vertices α' of $H(m - 1, q)$ such that there exists an $\alpha \in C$ with $\alpha_i = 0$ and $\alpha'_j = \alpha_j$ for all $j \in M \setminus \{i\}$. If the set of punctured (respectively shortened) codes obtained as i ranges over all possible coordinates are pairwise equivalent then we simply say the punctured (respectively shortened) code of C; in particular, this occurs if $\mathrm{Aut}(C)_{\mathbf{0}}$ acts transitively on M (for notation regarding automorphisms see Section 2.1). The *even-weight subcode* of C is the code consisting of precisely those codewords in C having even weight.

The *Singleton bound* (see [12, 4.3.2]) is a well known bound for the size of a code C in $H(m, q)$ with minimum distance δ, stating that $\lvert C\rvert \leqslant q^{m-\delta+1}$. For a linear code C this may be stated as $\delta^\perp - 1 \leqslant k \leqslant m - \delta + 1$, where k is the dimension of C, δ is the minimum distance of C and $\delta^\perp$ is the minimum distance of $C^\perp$.

Let C be a code and s be the external distance of C, that is, $s + 1$ is the number of non-zero terms in the *dual distance distribution* of C. It follows from [36, Theorem 4.1] that if C is

completely regular then s is equal to the covering radius ρ of C.

2.1 Automorphisms of a Hamming graph

The automorphism group $\mathrm{Aut}(\Gamma)$ of the Hamming graph is the semi-direct product $B \rtimes L$, where $B \cong \mathrm{Sym}(Q)^m$ and $L \cong \mathrm{Sym}(M)$ (see [9, Theorem 9.2.1]). Note that B and L are called the *base group* and the *top group*, respectively, of $\mathrm{Aut}(\Gamma)$. Since we identify Q_i with Q, we also identify $\mathrm{Sym}(Q_i)$ with $\mathrm{Sym}(Q)$. If $h \in B$ and $i \in M$ then $h_i \in \mathrm{Sym}(Q_i)$ is the image of the action of h in the entry $i \in M$. Let $h \in B$, $\sigma \in L$ and $\alpha \in V\Gamma$. Then h and σ act on α explicitly via:
$$\alpha^h = (\alpha_1^{h_1}, \dots, \alpha_m^{h_m}) \quad \text{and} \quad \alpha^\sigma = (\alpha_{1^{\sigma^{-1}}}, \dots, \alpha_{m^{\sigma^{-1}}}).$$
The automorphism group of a code C in $\Gamma = H(m,q)$ is $\mathrm{Aut}(C) = \mathrm{Aut}(\Gamma)_C$, the setwise stabiliser of C in $\mathrm{Aut}(\Gamma)$.

Let G be a group acting on a set Ω, ω be an element of Ω, and S be a subset of Ω. Then we use the following notation:

1. G_ω denotes the subgroup of G stabilising ω,

2. G_S denotes the setwise stabiliser of S in G,

3. $G_{(S)}$ denotes the point-wise stabiliser of S in G, and,

4. if G fixes S setwise then G^S denotes the subgroup of $\mathrm{Sym}(S)$ induced by G.

(For more background and notation on permutation groups see, for instance, [13].) In particular, let $X \leqslant \mathrm{Aut}(\Gamma)$. Then:

1. For $x \in X$, recall that $x = h\sigma$ where $h \in B$ and $\sigma \in L$. Then $x^M = \sigma$ denotes the permutation of M induced by x, and we write $X^M = \{x^M \mid x \in X\}$; we call X^M the *action of X on entries*. Note that a pre-image x of an element x^M of X^M need not fix any vertex of $H(m,q)$.

2. $K = X \cap B$ is the kernel $X_{(M)}$ of the action of X on entries and is precisely the subgroup of X fixing M point-wise.

3. If $i \in M$, then X_i denotes the subgroup of X stabilising the entry i and any $x \in X_i$ is of the form $h\sigma$ ($h \in B$ and $\sigma \in L$) where σ fixes $i \in M$. So $x = h\sigma$ induces the permutation $h_i \in \mathrm{Sym}(Q_i)$ on the alphabet Q_i. This defines a homomorphism from X_i to $\mathrm{Sym}(Q_i)$ and we denote the image of this homomorphism by $X_i^{Q_i}$. We refer to $X_i^{Q_i}$ as the *action on the alphabet*.

It is worth mentioning that coding theorists often consider more restricted groups of automorphisms, such as the group $\mathrm{PermAut}(C) = \{\sigma \mid h\sigma \in \mathrm{Aut}(C), h = 1 \in B, \sigma \in L\}$. The elements of this group are called *pure permutations* on the entries of the code.

Two codes C and C' in $H(m,q)$ are said to be *equivalent* if there exists some $x \in \mathrm{Aut}(\Gamma)$ such that $C^x = \{\alpha^x \mid \alpha \in C\} = C'$. Equivalence preserves many of the important properties in coding theory, such as minimum distance and covering radius, since $\mathrm{Aut}(\Gamma)$ preserves distances in $H(m,q)$.

2.2 s-Neighbour-transitive codes

This section presents preliminary results regarding (X, s)-neighbour-transitive codes, defined in Definition 1.1. The next results give certain 2-homogeneous and 2-transitive actions associated with an $(X, 2)$-neighbour-transitive code.

Proposition 2.1. *[16, Proposition 2.5] Let C be an (X, s)-neighbour-transitive code in $H(m, q)$ with minimum distance δ, where $\delta \geqslant 3$ and $s \geqslant 1$. Then for $\alpha \in C$ and $i \leqslant \min\{s, \lfloor \frac{\delta-1}{2} \rfloor\}$, the stabiliser X_α fixes setwise and acts transitively on $\Gamma_i(\alpha)$. In particular, the action of X_α on M is i-homogeneous.*

Proposition 2.2. *[16, Proposition 2.7] Let C be an $(X, 1)$-neighbour-transitive code in $H(m, q)$ with minimum distance $\delta \geqslant 3$ and $|C| > 1$. Then $X_i^{Q_i}$ acts 2-transitively on Q_i for all $i \in M$.*

The concept of a block design, introduced below, comes up frequently in coding theory. Recall that we identify vertices in $H(m, 2)$ with subsets of M so that a design as defined below can be identified with a code in $H(m, 2)$.

Definition 2.3. A t-(m, k, λ) *design* is a collection $\mathcal{D}$ of subsets of a set M of size m such that every t-subset of M is contained in exactly λ elements of $\mathcal{D}$. The elements of $\mathcal{D}$ are called *blocks*.

The following equations can be found, for instance, in [37]. Let $\mathcal{D}$ be a t-(m, k, λ) design with $|\mathcal{D}| = b$ blocks and let r be the number of blocks containing any given point. Then $mr = bk$, $r(k - 1) = \lambda(m - 1)$ and

$$b = \frac{m(m-1)\cdots(m-s+1)}{k(k-1)\cdots(k-s+1)}\lambda. \tag{2.1}$$

The definition below is required in order to state the remaining two results of this section. Recall that any s-neighbour-transitive code is s-regular code and any completely transitive code is completely regular.

Definition 2.4. Let C be a code in $H(m, q)$ with covering radius ρ, and s be an integer with $0 \leqslant s \leqslant \rho$. Then,

1. C is s-*regular* if, for each $i \in \{0, 1, \ldots, s\}$, each $k \in \{0, 1, \ldots, m\}$, and every vertex $\nu \in C_i$, the number $|\Gamma_k(\nu) \cap C|$ depends only on i and k, and,

2. C is *completely regular* if C is ρ-regular.

Lemma 2.5. *[16, Lemma 2.16] Let C be an (X, s)-neighbour transitive code in $H(m, q)$. Then C is s-regular. Moreover, if C has minimum distance $\delta \geqslant 2s$ and contains $\mathbf{0}$, then for each $k \leqslant m$ the set of codewords of weight k forms a q-ary s-(m, k, λ) design, for some λ.*

2.3 Completely transitive codes

This section contains mostly new preliminary results relating to the main concern of this paper, completely transitive codes. In particular, many of the results of this section assume Hypothesis 1. We believe the next result to be known, though we were unable to find an exact reference.

Lemma 2.6. *Let C be a completely transitive code on $H(m, q)$, let $X = \mathrm{Aut}(C)$ and suppose that $X_{\mathbf{0}}$ acts t-homogeneously, but not $(t + 1)$-homogeneously, on M. Then $\delta \leqslant 2t + 2$.*

Proof. Suppose, for a contradiction, that $\delta \geqslant 2t + 3$. Then $\lfloor \frac{\delta-1}{2} \rfloor \geqslant t + 1$, so that any two balls of radius $t + 1$ centered around distinct codewords are disjoint. Note that this also implies that the covering radius ρ of C is at least $t + 1$. By assumption, C is (X, ρ)-neighbour-transitive, and hence Proposition 2.1 implies that $X_{\mathbf{0}}$ acts at least $(t + 1)$-homogeneously on M, giving a contradiction. $\square$

The following relies on [26, Proposition 3.5] and gives more explicit information about a code satisfying Hypothesis 1.

Proposition 2.7. *Assume Hypothesis 1. Then each of the following hold:*

1. $X_{max} = T_{C_{max}} \rtimes X_{\mathbf{0}}$.

2. C_{max} *is* $(X_{max}, 2)$*-neighbour-transitive with* $\delta_{max} \geqslant 5$.

3. *The kernel* K *of the action of* X *on* M *is equal to the kernel of the action of* X_{max} *on* M, *that is,* $K = X \cap B = X_{max} \cap B = T_{C_{max}}$.

4. C_{max} *is an* $\mathbb{F}_2 X_{\mathbf{0}}$*-module*

5. $\dfrac{|C|}{|C_{max}|} = \dfrac{|X|}{|X_{max}|} = \dfrac{|X^M|}{|X_{\mathbf{0}}^M|}$.

Proof. We shall apply [26, Proposition 3.5], and so we first we argue that C satisfies the required conditions. Since C has minimum distance $\delta \geqslant 5$, it follows that C has covering radius at least 2. Hence, X acts transitively on C, C_1 and C_2, each of which are non-empty, so that C is $(X, 2)$-neighbour-transitive. By Proposition 2.1, X acts transitively on M, by Proposition 2.2 we have that $X_i^{Q_i} \cong S_2$, for each $i \in M$, and $T_{C_{max}} \leqslant X$ implies that $K \neq 1$. Hence C is X-*alphabet-affine* (see [26, Definition 1.4]) and we may apply [26, Proposition 3.5].

We claim that $O_2(K) = K = T_{C_{max}}$, where $K = X \cap B$ and $O_2(K)$ is the largest normal 2-subgroup of K. This implies that the code $W = \mathbf{0}^{O_2(K)}$ appearing in the statement of [26, Proposition 3.5] is in fact C_{max}, and that the above parts 1, 2, 3 and 4 hold. Since $T_{C_{max}}$ is a 2-group and hence $O_2(T_{C_{max}}) = T_{C_{max}}$, in order to prove the claim we need only show that $K = T_{C_{max}}$. Suppose $x \in K$. Then $x = (h_1, \ldots, h_m)$ and each $h_i \in \mathbb{Z}_2$, that is, $x = t_\alpha$ where α is the vertex of $H(m, 2)$ such that $\alpha_i = 0$ if $h_i = 0$ and $\alpha_i = 1$ if $h_i = 1$, for each $i \in M$. It follows that there exists a subcode C' of C such that $K = T_{C'}$. Since K is a group and $q = 2$ (and hence $\mathbb{F}_2^\times$ is the trivial group) we have that C' is linear. The fact $K = X \cap B$ implies that there is no larger group of translations contained in X and thus, by Definition 1.4, we have that $C' = C_{max}$ and $K = T_{C_{max}}$, proving the claim.

Now $K = T_{C_{max}}$ is normal in X, which means that the $T_{C_{max}}$-orbits of C form a system of imprimitivity for the action of X on C. Since X acts transitively on the set of $T_{C_{max}}$-orbits of C, the orbit-stabiliser theorem implies that

$$\frac{|C|}{|C_{max}|} = \frac{|X|}{|X_{max}|} = \frac{|X/T_{C_{max}}|}{|X_{max}/T_{C_{max}}|} = \frac{|X^M|}{|X_{\mathbf{0}}^M|},$$

where the final equality follows from parts 1 and 3. Thus, part 5 holds. $\square$

The following result shows, roughly speaking, that if a completely transitive code is not much bigger than its maximal linear subcode then the minimum distance of the maximal linear subcode is not much bigger than the minimum distance of the code. When combined with Lemma 2.6, this leads to non-existence results in several cases.

Lemma 2.8. *Assume Hypothesis 1 holds. If*

$$\frac{|C|}{|C_{max}|} < \frac{m(m-1)}{\delta(\delta-1)},$$

then $\delta_{max} \leqslant 2\delta$.

Proof. We shall prove the contrapositive, and hence we assume that $\delta_{\max} \geqslant 2\delta + 1$ and will prove that then $\frac{|C|}{|C_{\max}|} \geqslant \frac{m(m-1)}{\delta(\delta-1)}$. By Lemma 2.5, the set $C \cap \Gamma_\delta(\mathbf{0})$ of weight δ codewords of C form a 2-(m, δ, λ) design, for some integer λ. Hence

$$|C \cap \Gamma_\delta(\mathbf{0})| = \frac{m(m-1)\lambda}{\delta(\delta-1)} \geqslant \frac{m(m-1)}{\delta(\delta-1)}.$$

For $\alpha, \beta \in C \cap \Gamma_\delta(\mathbf{0})$ we have, by the triangle inequality, that $d(\alpha, \beta) \leqslant 2\delta$. Since $\delta_{\max} \geqslant 2\delta+1$, it follows that no two elements of $C \cap \Gamma_\delta(\mathbf{0})$ are contained in the same coset of $C_{\max}$. Thus, C is comprised of at least $|C \cap \Gamma_\delta(\mathbf{0})|$ cosets of $C_{\max}$ and the result holds. □

The following lemma is useful for proving non-existence of X-neighbour-transitive codes in the case where we are dealing with a family of groups X having order that is polynomial in the length m of the code.

Lemma 2.9. *Let C be an X-completely transitive code in $H(m, q)$. Then*

$$(m+1)|X| \geqslant q^m.$$

Proof. Since C is X-completely transitive, we have that there are $\rho + 1$ orbits of X on the vertex set of $H(m, q)$. Now, the covering radius ρ is bounded above by the diameter m of $H(m, q)$, and hence $m + 1 \geqslant \rho + 1$. Thus, using the fact that the length of any orbit of X on vertices of $H(m, q)$ is bounded above by $|X|$, we have that

$$(m+1)|X| \geqslant (\rho+1)|X| \geqslant |V\Gamma| = q^m,$$

and the result holds. □

The next result shows that any completely transitive code in $H(m, 2)$, where Q is taken to be $\mathbb{F}_2$, is a subcode of some linear completely transitive code.

Lemma 2.10. *Assume Hypothesis 1. Furthermore, suppose $C \neq C_{max}$, let n be the integer such that $|C| = n|C_{max}|$ and let $\langle C \rangle$ be the $\mathbb{F}_2$-span of the codewords of C. Then each of the following holds:*

1. $\mathrm{Aut}(C) \leqslant \mathrm{Aut}(\langle C \rangle)$.

2. $\langle C \rangle$ *is completely transitive.*

3. *The codimension of C_{max} in $\langle C \rangle$ is at least 2 and at most $n-1$.*

Proof. Let $\alpha \in \langle C \rangle$. Then there exists an integer k such that $\alpha = \alpha_1 + \cdots + \alpha_k$, where each $\alpha_i \in C$. Let $x = h\sigma \in \mathrm{Aut}(C)$, where $h \in B \cong \mathbb{Z}_2^m$ and $\sigma \in L$, and let $\beta = \mathbf{0}^h$. Note that $\beta \in C$ since σ, and hence σ^{-1}, fixes the vertex $\mathbf{0}$, so that

$$\mathbf{0}^{x^{-1}} = \mathbf{0}^{\sigma^{-1}h^{-1}} = \mathbf{0}^{h^{-1}} = \beta,$$

with the last step following from the fact that $h = h^{-1}$. Hence $x = t_\beta \sigma$, where t_β is the translation by β. Now $\mathrm{PermAut}(\Gamma) = \mathrm{S}_m$ embeds as the group of permutation matrices in $\mathrm{GL}_m(2)$, from which it follows that $V\Gamma$ is an $\mathbb{F}_2 \,\mathrm{PermAut}(\Gamma)$-module. Since $\sigma \in \mathrm{PermAut}(\Gamma)$, we have that

$$\alpha^x = (\alpha + \beta)^\sigma = (\alpha_1)^\sigma + \cdots + (\alpha_k)^\sigma + \beta^\sigma$$
$$= (\alpha_1 + \beta)^x + \cdots + (\alpha_k + \beta)^x + \mathbf{0}^x.$$

As $\mathbf{0}$ and each $\alpha_i + \beta$ are in C, and $x \in \mathrm{Aut}(C)$, we have that $\mathbf{0}^x$ and each $(\alpha_i + \beta)^x$ are also in C. Thus, we deduce that $\alpha^x \in \langle C \rangle$, which implies that $x \in \mathrm{Aut}(\langle C \rangle)$. Thus $\mathrm{Aut}(C) \leqslant \mathrm{Aut}(\langle C \rangle)$.

Since C is a subcode of $\langle C \rangle$, it follows that the covering radius ρ' of $\langle C \rangle$ is at most the covering radius ρ of C. First, note that $\mathrm{Aut}(\langle C \rangle)$ contains the translations by $\langle C \rangle$ and thus acts transitively on $\langle C \rangle$. Let $\mu, \nu \in \langle C \rangle_i$ for some i, where $1 \leqslant i \leqslant \rho'$. Then there exist $\alpha, \beta \in \langle C \rangle$ such that $d(\alpha, \mu) = d(\beta, \nu) = i$. If t_α and t_β are the translations by α and β, respectively, then it follows that $\alpha^{t_\alpha} = \mathbf{0}$ and $\beta^{t_\beta} = \mathbf{0}$ so that $d(\mathbf{0}, \mu^{t_\alpha}) = d(\mathbf{0}, \nu^{t_\alpha}) = i$. Hence, since $\mathbf{0} \in C$, we have that μ^{t_α} and ν^{t_β} are in C_i, and thus there exists an $x \in \mathrm{Aut}(C)$ such that $(\mu^{t_\alpha})^x = \nu^{t_\beta}$. This implies that $\mu^{t_\alpha x t_\beta} = \nu$ and, since $\mathrm{Aut}(C) \leqslant \mathrm{Aut}(\langle C \rangle)$, that $\mathrm{Aut}(\langle C \rangle)$ acts transitively on $\langle C \rangle_i$. Thus $\langle C \rangle$ is completely transitive, proving part 2.

By Theorem 1.5 part 3 (b), C is the union of $C_{\max}$ and $n - 1$ non-trivial cosets of $C_{\max}$. Let $v_1, \ldots, v_{n-1}$ be coset representatives for these $n - 1$ non-trivial cosets. It follows that $\langle C \rangle = \langle C_{\max}, v_1, \ldots, v_{n-1} \rangle$, and hence that the codimension of $C_{\max}$ in $\langle C \rangle$ is at most $n - 1$. Suppose that the codimension of $C_{\max}$ in $\langle C \rangle$ is 1 and hence, since $q = 2$, we have that $n = 2$. Thus $|C| = 2|C_{\max}|$ and $\langle C \rangle = \langle C_{\max}, v_1 \rangle$, that is $|\langle C \rangle| = 2|C_{\max}|$, and in fact $C = \langle C \rangle$. This implies that C is linear, contradicting the assumption that $C \neq C_{\max}$. Thus the codimension of $C_{\max}$ in $\langle C \rangle$ is at least 2, as in part 3. $\qquad\square$

The following result effectively gives upper bounds for the number of weight δ codewords in a binary completely transitive code when δ is 5 or 6.

Lemma 2.11. *Let C be a completely transitive code in $H(m, 2)$ with minimum distance $\delta = 5$ or 6. Then the set of all weight δ codewords form a t-(m, δ, λ) design, where:*

1. *$\delta = 5$, $t = 2$ and $\lambda \leqslant (m - 2)/3$, or;*

2. *$\delta = 6$, $t = 3$ and $\lambda \leqslant (m - 3)/3$.*

Proof. The set of all weight δ codewords form the block set $\mathcal{B}$ of a t-(m, δ, λ) design by [8, Theorem 2.12], where $t = 2$ when $\delta = 5$ and $t = 3$ when $\delta = 6$. First, let $\delta = 5$ and i, j be distinct elements of M. Let $\mathcal{B}_{i,j}$ be the set of blocks incident with i and j projected down to $M \setminus \{i, j\}$. It follows that there are λ elements of $\mathcal{B}_{i,j}$, each one being a subset of size 3 in $M \setminus \{i, j\}$. Let α and β be distinct subsets of M of size 5 with $i, j \in \alpha, \beta$. Then $d(\alpha, \beta)$ is either 2, 4 or 6 depending on whether $\alpha \cap \beta$ has size 4, 3 or 2, respectively. Thus, since $\delta = 5$, if we assume α and β are in C, then $|\alpha \cap \beta| = 2$ and the images of α and β in $\mathcal{B}_{i,j}$ are disjoint. Since any two distinct blocks of $\mathcal{B}_{i,j}$ are disjoint there are at most $(m - 2)/3$ such blocks, and hence at most $(m - 2)/3$ weight 5 codewords that are non-zero in a particular pair of coordinates, proving part 1.

Let $\delta = 6$ and i, j, k be pairwise distinct elements of M. Let $\mathcal{B}_{i,j,k}$ be the set of blocks incident with i, j and k projected down to $M \setminus \{i, j, k\}$. Again, there are λ elements of $\mathcal{B}_{i,j,k}$, and each block is incident with 3 elements of $M \setminus \{i, j, k\}$. Let α and β be distinct subsets of M of size 6 with $i, j, k \in \alpha, \beta$. Then $d(\alpha, \beta)$ is either 2, 4 or 6 depending on whether $\alpha \cap \beta$ has size 5, 4 or 3, respectively. Thus, since $\delta = 6$, if we assume α and β are in C, then $|\alpha \cap \beta| = 3$

and the images of α and β in $\mathcal{B}_{i,j,k}$ are disjoint. Since any two distinct blocks of $\mathcal{B}_{i,j,k}$ are disjoint there are at most $(m-3)/3$ such blocks, and hence at most $(m-3)/3$ weight 6 codewords that are non-zero in a particular triple $\{i,j,k\}$ of coordinates, proving part 2. $\qquad\square$

3   The Codes

In this section we fix our notation for the codes that arise in later sections, giving explicit constructions where required, references and other details required in later sections. From this point forward, all codes considered are binary. Recall that for binary codes we will be flexible with our interpretation of the vertices of $H(m,2)$; see Section 2. More specifically, we interchangeably view vertices as: binary strings indexed by M, subsets of M, or functions from M to $Q = \{0,1\}$.

Example 3.1. [16, Definition 4.1] We obtain the Hadamard 12 code $\mathcal{H}$ and its punctured code $\mathcal{PH}$ as follows (see [32, Part 1, Section 2.3]). First, we construct a normalised Hadamard matrix H_{12} of order 12 using the Paley construction.

1. Let $M = \mathbb{F}_{11} \cup \{*\}$ and let H_{12} be the 12×12 matrix with first row v, where $v_a = -1$ if a is a square in $\mathbb{F}_{11}$ (including 0), and $v_a = 1$ if a is a non-square in $\mathbb{F}_{11}$ or $a = * \in M$, taking the orbit of v under the additive group of $\mathbb{F}_{11}$ acting on M to form 10 more rows and adding a final row, the vector $(-1,\ldots,-1)$.

2. The Hadamard code $\mathcal{H}$ of length 12 in $H(12,2)$ then consists of the vertices α such that there exists a row u in H_{12} or $-H_{12}$ satisfying $\alpha_a = 0$ when $u_a = 1$ and $\alpha_a = 1$ when $u_a = -1$.

3. The punctured code $\mathcal{PH}$ of $\mathcal{H}$ is obtained by deleting the coordinate $*$ from M. The weight 6 codewords of $\mathcal{PH}$ form a binary 2-$(11,6,3)$ design $\mathcal{D}$. The code $\mathcal{PH}$ consists of the following codewords: the zero codeword, the vector $(1,\ldots,1)$, the characteristic vectors of the 2-$(11,6,3)$ design $\mathcal{D}$, and the characteristic vectors of the complement of that design, which forms a 2-$(11,5,2)$ design. (Both $\mathcal{D}$ and its complement are unique up to isomorphism [39].)

4. The even weight subcode $\mathcal{E}$ of $\mathcal{PH}$ is the code consisting of the zero codeword and the 2-$(11,6,3)$ design.

Example 3.2. The Nordstrom–Robinson code, which we denote throughout by $\mathcal{NR}$ is a well-known non-linear code having parameters $(15,256,5;3)$. Since we will not require a precise definition of the Nordstrom–Robinson code here, we refer the reader to [22] or [24], suffice it to say that $\mathcal{NR}$ is the union of 8 cosets of the Reed–Muller code $\mathcal{RM}(1,4)$ and is contained in $\mathcal{RM}(2,4)$.

Example 3.3. Let M be the set of points of $\mathrm{PG}_2(4)$. We define the following codes:

1. Let $\mathcal{P}$ be the code generated by the set of all complements of lines of $\mathrm{PG}_2(4)$ and $\mathcal{L}$ be the code generated by the set of all lines of $\mathrm{PG}_2(4)$.

2. The group $\mathrm{PSL}_3(4)$ has three orbits $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ on hyperovals (*i.e.* sets of $q+2$ points with no three collinear), and three orbits $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ on Fano subplanes labelled so that if $\Delta \in \mathcal{H}_i$ and $\Phi \in \mathcal{F}_j$ then $|\Delta \cap \Phi| \leqslant 3$ if and only if $i \neq j$ (see [13, Theorem 6.5B] and also Example 3.4). Let ℓ be a line of $\mathrm{PG}_2(4)$ and, for each $i = 1,2,3$, let $\Phi_i \in \mathcal{F}_i$ and let $\Delta_i \in \mathcal{H}_i$. Let $R = \{\ell, \Delta_1, \Delta_2, \Delta_3, \Phi_1, \Phi_2, \Phi_3\}$. For a given code C in $H(21,2)$ and

some $\alpha_1, \ldots, \alpha_j \in R$ we shall often consider the code given by the linear span of C and $\alpha_1, \ldots, \alpha_j$ over $\mathbb{F}_2$, which we denote simply by

$$\langle C, \alpha_1, \ldots, \alpha_j \rangle.$$

For example, we show in Section 6 that $\langle \mathcal{P}, \ell \rangle = \mathcal{L}$. The codes arising in this manner and appearing in Table 2 are $\langle \mathcal{L}, \Delta_1 \rangle$ and $\langle \mathcal{L}, \Delta_1 \rangle \cup \langle \mathcal{L}, \Delta_2 \rangle$.

Example 3.4. Let $\{a, b, c\}$ be a set of size 3 that is disjoint from the set P of points of $\mathrm{PG}_2(4)$ and let $M = P \cup S$, where S is a subset of $\{a, b, c\}$ of size 1, 2 or 3. Thus $m = 22$, 23 or 24. Following [13, Theorem 6.7C and Section 6.5-6.7], fix a construction for the block set of the Witt design W_{24}, which will correspond to the set of weight 8 codewords of the extended binary Golay code $\mathcal{G}_{24}$ in $H(24, 2)$. The group $\mathrm{PSL}_3(4)$ has three orbits $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ on hyperovals (sets of $q + 2$ points, no three collinear), and three orbits $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ on Fano subplanes labelled so that for $\Delta \in \mathcal{H}_i$ and $\Phi \in \mathcal{F}_j$ we have that $|\Delta \cap \Phi| \leqslant 3$ if and only if $i \neq j$. The blocks of W_{24} are then:

1. $\ell \cup \{a, b, c\}$ for each line ℓ of $\mathrm{PG}_2(4)$,

2. $\Delta \cup \{a, b\}$ for each hyperoval $\Delta \in \mathcal{H}_1$,

3. $\Delta \cup \{b, c\}$ for each hyperoval $\Delta \in \mathcal{H}_2$,

4. $\Delta \cup \{c, a\}$ for each hyperoval $\Delta \in \mathcal{H}_3$,

5. $\Phi \cup \{c\}$ for each Fano plane $\Phi \in \mathcal{F}_1$,

6. $\Phi \cup \{a\}$ for each Fano plane $\Phi \in \mathcal{F}_2$,

7. $\Phi \cup \{b\}$ for each Fano plane $\Phi \in \mathcal{F}_3$,

8. the symmetric difference $\ell_1 + \ell_2$ for each pair of distinct lines of $\mathrm{PG}_2(4)$.

We define the following codes:

1. For $S = \{a, b, c\}$, and hence $m = 24$, we denote the code spanned by the blocks of W_{24} by $\mathcal{G}_{24}$, which is self-dual.

2. For $S = \{a, b\}$, and hence $m = 23$, we denote the punctured code of $\mathcal{G}_{24}$ by $\mathcal{G}_{23}$ and its dual, the shortened code of $\mathcal{G}_{24}$, by $\mathcal{G}_{23}^{\perp}$.

3. For $S = \{a\}$, and hence $m = 22$, we denote the punctured code of $\mathcal{G}_{23}$ by $\mathcal{G}_{22}$, the shortened code of $\mathcal{G}_{23}$ by $\mathcal{S}_{22}$ and the even weight subcode of $\mathcal{G}_{22}$ by $\mathcal{E}_{22}$.

Remark 3.5. Note that there is a typographical error on page 204 of [13] in the list of the different types of blocks in W_{24} (v) and (vii): $\mathcal{F}_1$ and $\mathcal{F}_3$ should be interchanged. It is not difficult to see that this is required by the condition that if $\Delta \in \mathcal{H}_i$ and $\Phi \in \mathcal{F}_j$ then $|\Delta \cap \Phi| \leqslant 3$ if and only if $i \neq j$. Also, [13, Theorem 6.5B (iv)] claims that $\mathrm{P\Gamma L}_3(4)$ induces a cyclic permutation on the set $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ of $\mathrm{PSL}_3(4)$-orbits of hyperovals. In particular, the proof of [13, Theorem 6.5B (iv)] claims without justification that the quotient $\mathrm{P\Gamma L}_3(4)/\mathrm{PSL}_3(4)$ is cyclic of order 6. Note that this is not true; indeed we show in Lemma 5.3 that $\mathrm{P\Gamma L}_3(4)/\mathrm{PSL}_3(4)$ is isomorphic to S_3.

3.1 Distance-regular graphs

The standard method (see [9, Chapter 11]) of constructing a distance-regular graph from completely regular code C in a distance-regular graph Γ is to use C (in some way) to determine a *completely regular* partition Π of the vertex set of Γ. In particular, each element of Π must be a completely regular code with the same *intersection numbers* as C. Given such a distance-regular partition Π of Γ we have that, by [9, Theorem 11.1.6], the quotient graph Γ/Π is distance-regular.

Suppose C is a linear completely regular code in a Hamming graph $H(m, q)$ with a group T of translations acting transitively on the vertex set of $H(m, q)$. Then the group T_C of all translations by elements of C is a normal subgroup of T and hence the set of orbits of T_C form a system of imprimitivity Π for the action of T on the vertex set of $H(m, q)$ and, by [9, Theorem 11.1.6], Π is completely regular. The graph $H(m, q)/\Pi$ constructed from C in this manner is called the *coset graph* of C. Hence, all of the linear codes from Table 2 give rise to (known) distance-regular coset graphs in this manner. A reference for each is given in the relevant line of Table 2. Note that $\mathcal{G}_{23}^{\perp}$ and $\mathcal{E}_{22}$ are the even weight subcodes of $\mathcal{G}_{23}$ and $\mathcal{G}_{22}$, respectively, so that their coset graphs are the bipartite doubles of the coset graphs of $\mathcal{G}_{23}$ and $\mathcal{G}_{22}$.

The question of whether a given non-linear completely regular code can be used to construct a distance-regular graph is more complicated. By [9, Proposition 11.1.5 (i)], each part of a completely regular partition Π involving a completely regular code C must have the same size as C. Thus, such a construction requires that the number of vertices of $H(m, q)$ is a multiple of $|C|$. Now, the codes $\mathcal{H}$, $\mathcal{PH}$ and $\langle\mathcal{L}, \Delta_1\rangle \cup \langle\mathcal{L}, \Delta_2\rangle$ each have size divisible by an odd prime, and hence these codes do not give rise to distance-regular graphs as quotients of $H(m, 2)$. From Table 2, this leaves the code $\mathcal{NR}$, for which we outline a construction in the following paragraph. Note that while the construction appears to be known, it is not explicitly described in the literature, as far as the authors are aware.

The Preparata codes (see [1]) of length $2^{2t} - 1$ are an infinite family of completely regular codes, of which the Nordstrom–Robinson code $\mathcal{NR}$ is the smallest, given by setting $t = 2$. It should be noted that although $\mathcal{NR}$ is the unique code when $t = 2$, there are several non-equivalent Preparata codes for each $t \geqslant 3$. In [1, Section III], for all $t \geqslant 2$, a partition of the extended Hamming code $\bar{H}$ of length 2^{2t} by translates of the extended Preparata code $\bar{C}$ of length 2^{2t} is provided, which (upon puncturing) carries over to a partition of the perfect Hamming code H of length $2^{2t} - 1$ by translates of the Preparata code C of the same length. Note that $H = C \cup C_3$, which means that in order to construct the required partition Π it remains to partition the set of all vertices of $H(2^{2t} - 1, 2)$ not contained in H. Since the Preparata codes have minimum distance 5, we may thus construct $\mathcal{P}$ as the union of i) the partition of H just described with ii) the set of all translates of C by a vertex of weight 1 or 2. Each translate of the Preparata code is an equivalent code, so is completely regular with the same intersection numbers, and thus the graph $\Sigma = H(2^{2t} - 1, 2)/\Pi$ is distance-regular. The intersection array of Σ is $\{2^{2t} - 1, 2^{2t} - 2, 1; 1, 2, 2^{2t} - 1\}$, which implies that Σ is a 2^{2t-1}-antipodal cover of the complete graph $\mathrm{K}_{2^{2t}}$.

4 A_7

In this section we assume that Hypothesis 1 holds, and that the socles of X/K and $X_{\max}/K$ are distinct. As we shall see, this occurs in precisely one case: when $X_{\max}/K \cong \mathrm{A}_7$ and C is equivalent to the Nordstrom–Robinson code.

Lemma 4.1. *Assume Hypothesis 1. Then* $\mathrm{soc}(X/K) \neq \mathrm{A}_m$.

Proof. Suppose that $\mathrm{soc}(X/K) = \mathrm{A}_m$, in which case we have that $X^M \cong X/K \cong \mathrm{A}_m$ or S_m.

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Hypothesis 1 implies that C falls under part 3 of Theorem 1.5. Thus, it follows from [26, Table 1] that X_0^M does not contain A_m. By Proposition 2.1, we have that X_0^M is 2-homogeneous, and hence also primitive. Thus, by [40, Theorem 14.2], we have that

$$|\operatorname{S}_m : X_0^M| \geqslant \left\lfloor \frac{m+1}{2} \right\rfloor!.$$

Now $|\operatorname{S}_m : X_0^M| \leqslant 2|X^M : X_0^M|$ and hence, by part 4 of Proposition 2.7, we have that

$$\left\lfloor \frac{m+1}{2} \right\rfloor! \leqslant 2\frac{|C|}{|C_{\mathsf{max}}|}.$$

Since C_{max} has dimension at least 2, and thus size at least 4, we deduce that

$$\left\lfloor \frac{m+1}{2} \right\rfloor! \leqslant |C|/2.$$

Since $\delta \geqslant 5$, the Singleton bound implies that $|C| \leqslant 2^{m-4}$. Combining the last two inequalities gives

$$\left\lfloor \frac{m+1}{2} \right\rfloor! \leqslant 2^{m-5}.$$

However, this does not hold for $m \geqslant 5$, and hence $\operatorname{soc}(X/K) \neq A_m$. $\qquad\square$

Lemma 4.2. *Assume Hypothesis 1 holds, and suppose that* $\operatorname{soc}(X_{max}/K) \neq \operatorname{soc}(X/K)$. *Then* $m = 15$, $X_{max}/K \cong A_7$ *and* $X/K \cong A_8$.

Proof. By Proposition 2.1, X_0^M is 2-homogeneous on M and, by Proposition 2.7, we have that $X_0^M \cong X_{\mathsf{max}}/K$. By [30], any 2-homogeneous but not 2-transitive group of degree m is a subgroup of $\operatorname{A\Gamma L}_1(m)$, that is, has socle the additive group $\mathbb{F}_m^+$ of the field $\mathbb{F}_m$. If X_0^M is 2-homogeneous but not 2-transitive then, since $\operatorname{soc}(X/K) \neq \operatorname{soc}(X_{\mathsf{max}}/K)$, it follows that the socle of $X/K \cong X^M$ is not $\mathbb{F}_m^+$, and hence that X acts 2-transitively on M. Hypothesis 1 requires that C_{max} has dimension at least 2, and hence Theorem 1.5 implies that $\operatorname{soc}(X_0)$ is as in one of the lines of [26, Table 1]. By Lemma 4.1, we have that $\operatorname{soc}(X/K) \neq A_m$. Thus, we may apply [16, Proposition 4.4], in which case X_0 is one of the groups G in [16, Table 3]. Those groups $G = X_0$ such that $\operatorname{soc}(X_0)$ appears in [26, Table 1] and G appears in [16, Table 3] (thus giving the corresponding $H \cong X/K$) leave us with the possibility that either:

1. $m = 15$, $X_0 \cong A_7$ and $X/K \cong A_8$;

2. $m = 23$, $X_0 \cong \mathbb{Z}_{23} \rtimes \mathbb{Z}_{11}$ and $X/K \cong M_{23}$; or,

3. $m = 24$, $X_0 \cong \operatorname{PSL}_2(23)$ and $X/K \cong M_{24}$.

Suppose $m = 23$. Then by part 4 of Proposition 2.7,

$$|C| = |X^M : X_0^M| \cdot |C_{\mathsf{max}}| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot |C_{\mathsf{max}}|.$$

In particular, $|C| \geqslant 2^{14}|C_{\mathsf{max}}|$. By [26, Table 1], $|C_{\mathsf{max}}| \geqslant 2^{11}$ so that $|C| \geqslant 2^{25}$. However, this is greater than the total number of vertices in $H(23, 2)$, giving a contradiction. Suppose $m = 24$. Then by part 4 of Proposition 2.7,

$$|C| = 2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot |C_{\mathsf{max}}|.$$

In particular, $|C| \geqslant 2^{14}|C_{\mathsf{max}}|$. By [26, Table 1], $|C_{\mathsf{max}}| \geqslant 2^{12}$ so that $|C| \geqslant 2^{26}$. However, this is greater than the number of vertices in $H(24, 2)$, giving a contradiction. Thus $m = 15$, $X_{\mathsf{max}}^M \cong A_7$ and $X^M \cong A_8$. $\qquad\square$

This brings us to the first of our classification results.

Proposition 4.3. *Suppose Hypothesis 1 holds, $m = 15$ and $X_{max}/K \cong A_7$. Then C is the Nordstrom–Robinson code $\mathcal{NR}$ in $H(15, 2)$ and $\delta = 5$.*

Proof. Suppose $\delta = 5$. Then, by [22, Theorem 1.1], C is equivalent to the Nordstrom–Robinson code. Thus $\delta \geqslant 6$. Now A_7 acts 2-transitively, but not 3-homogeneously, on 15 points so that, by Lemma 2.6, $\delta = 6$. Thus, by [2, Table I], $|C| \leqslant 128$. Also, by Theorem 1.5 and [26, Table 1, line 4], the minimal $(X, 2)$-neighbour-transitive subcode of C has minimum distance 8, and hence C contains codewords of weight 8. By [8, Theorem 2.12], the set of all weight k codewords form a 3-design, so that, for some integers $\lambda_6, \lambda_8 \geqslant 1$, we have:

$$|C \cap \Gamma_6(\mathbf{0})| = \frac{7 \cdot 13}{4}\lambda_6 \quad \text{and} \quad |C \cap \Gamma_8(\mathbf{0})| = \frac{5 \cdot 13}{8}\lambda_8.$$

Since each of these must be integers, we have that 4 divides λ_6 and 8 divides λ_8. Hence $|C \cap \Gamma_6(\mathbf{0})| \geqslant 91$ and $|C \cap \Gamma_8(\mathbf{0})| \geqslant 65$. However, this implies that $|C| \geqslant 91 + 65$, contradicting $|C| \leqslant 128$. □

5 The Mathieu Groups

Throughout this section we assume that Hypothesis 2 holds. In particular, now $\mathrm{soc}(X/K) = \mathrm{soc}(X_{\max}/K)$. Moreover, we assume that $\mathrm{soc}(X/K) \cong \mathrm{M}_m$, where $m = 22$, 23 or 24. For the constructions of the codes of this section, and an explanation of the notation used, see Example 3.4.

We now determine the possibilities for $C_{\max}$, which must be a submodule of the permutation module over $\mathbb{F}_2$ for M_m and, since $2 \leqslant \dim(C_{\max}) \leqslant m - 2$, not equal to either $\langle \mathbf{1} \rangle$ or $\langle \mathbf{1} \rangle^\perp$. For $m = 23$ and 24 this is covered in the discussion at the beginning of Section 8 of [29], with only partial information given there for $m = 22$. Code parameters can be found in [8, Section 4.1]. If $m = 24$ then $C_{\max}$ is the extended binary Golay code $\mathcal{G}_{24}$ with parameters $[24, 12, 8; 4]$. If $m = 23$, then $C_{\max}$ contains the dual code $\mathcal{G}_{23}^\perp$ of the binary Golay code $\mathcal{G}_{23}$. Hence $C_{\max}$ is either $\mathcal{G}_{23}$, which has parameters $[23, 12, 7; 3]$, or $\mathcal{G}_{23}^\perp$, which has parameters $[23, 11, 8; 7]$. Note that if M is then taken to be the points of $\mathrm{PG}_2(4)$ with a and b (see Example 3.4) adjoined then $\mathcal{G}_{23}$ may be obtained as the linear span of the blocks of W_{24} containing the point c, and $\mathcal{G}_{23}^\perp$ may be obtained as the linear span of the blocks not containing c.

Let $m = 22$ and $\mathcal{G}_{22}$ denote the punctured binary Golay code. Then $C_{\max}$ contains $\mathcal{G}_{22}^\perp$, which has parameters $[22, 10, 8; 7]$, and is contained in $\mathcal{G}_{22}$, which has parameters $[22, 12, 6; 3]$. Letting M be the points of $\mathrm{PG}_2(4)$ with a adjoined (see Example 3.4), the linear span of the set of those blocks of W_{24} containing both b and c is $\mathcal{G}_{22}$, and the linear span of the set of blocks containing neither b nor c is $\mathcal{G}_{22}^\perp$. The linear span of the set of blocks containing neither b nor c along with the vector $(1, 1, \ldots, 1)$ gives the even weight subcode $\mathcal{E}_{22}$ of $\mathcal{G}_{22}$, which has parameters $[22, 11, 6; 7]$. Two further (equivalent) codes, each with parameters $[22, 11, 7; 6]$, are obtained by taking the linear span of either the blocks containing b but not c, or the blocks containing c but not b, one of which is the shortened code $\mathcal{S}_{22}$ of $\mathcal{G}_{23}$. Each of $\mathcal{G}_{22}$, $\mathcal{G}_{22}^\perp$ and $\mathcal{E}_{22}$ are invariant under $\mathrm{Aut}(\mathrm{M}_{22}) = \mathrm{Aut}(W_{24})_{\{b,c\}} = \mathrm{M}_{22} : 2 \leqslant X_0^M$, whilst the outer automorphism of $\mathrm{M}_{22} = \mathrm{Aut}(W_{24})_{b,c}$ interchanges the two $[22, 11, 7; 6]$ codes. These are all the codes for $m = 22$, since there are three non-trivial cosets of $\mathcal{G}_{22}^\perp$ in $\mathcal{G}_{22}$. Thus, we have shown the following.

Lemma 5.1. *Let $m = 22, 23$ or 24, let C be a linear code in $H(m, 2)$ with $2 \leqslant \dim(C) \leqslant m - 2$, let $X = \mathrm{Aut}(C)$ and suppose $\mathrm{soc}(X/K) \cong \mathrm{M}_m$. Then $C = C_{max}$ is equivalent to one of following codes in $H(m, 2)$.*

- *The $[24, 12, 8; 4]$ extended binary Golay code $\mathcal{G}_{24}$.*

- *The $[23, 12, 7; 3]$ binary Golay code $\mathcal{G}_{23}$.*

- *The $[23, 11, 8; 7]$ dual $\mathcal{G}_{23}^{\perp}$ of $\mathcal{G}_{23}$.*

- *The $[22, 12, 6; 3]$ punctured binary Golay code $\mathcal{G}_{22}$.*

- *The $[22, 11, 6; 7]$ even weight subcode $\mathcal{E}_{22}$ of $\mathcal{G}_{22}$.*

- *The $[22, 11, 7; 6]$ shortened binary Golay code $\mathcal{S}_{22}$.*

- *The $[22, 10, 8; 7]$ dual code $\mathcal{G}_{22}^{\perp}$.*

In the remainder of this section we determine all the completely transitive codes that arise in this case, each of which is ijn fact a code from the previous lemma.

Lemma 5.2. *Suppose $m = 23$ or 24 and Hypothesis 2 holds with $\mathrm{soc}(X/K) \cong \mathrm{M}_m$. Then C is equivalent to one of following completely transitive codes in $H(m, 2)$:*

- *The $[24, 12, 8; 4]$ extended binary Golay code $\mathcal{G}_{24}$.*

- *The $[23, 12, 7; 3]$ binary Golay code $\mathcal{G}_{23}$.*

- *The $[23, 12, 8; 7]$ dual code $\mathcal{G}_{23}^{\perp}$.*

Proof. Since we are assuming $\mathrm{soc}(X/K) = \mathrm{soc}(X_{\mathsf{max}}/K)$ and we know that $\mathrm{Aut}(\mathrm{M}_m) \cong \mathrm{M}_m$ for $m = 23$ and 24, we have, by Proposition 2.7 part 4, that $C = C_{\mathsf{max}}$. Thus, by Lemma 5.1, C is either $\mathcal{G}_{24}$, $\mathcal{G}_{23}$ or $\mathcal{G}_{23}^{\perp}$. Both $\mathcal{G}_{24}$ and $\mathcal{G}_{23}$ are completely transitive, by [36, Example on p. 199].

Let $C = \mathcal{G}_{23}^{\perp}$ which, as noted in the discussion preceding Lemma 5.1, may be obtained as the linear span of the blocks of W_{24} not containing c, where M is taken to be the points of $\mathrm{PG}_2(4)$ with a and b adjoined. In particular, $\Delta \cup \{a, b\}$ is a codeword, for each hyperoval $\Delta \in \mathcal{H}_1$. Since T_C acts transitively on C and C has covering radius 7, it is enough to prove that M_{23} acts transitively on $C_i \cap \Gamma_i(\mathbf{0})$ for each $i = 1, \ldots, \rho = 7$. First, M_{23} acts 4-transitively on M, and thus transitively on $\Gamma_i(\mathbf{0})$ for $i = 1, 2, 3, 4$.

Consider a vertex $\alpha \in C_i \cap \Gamma_i(\mathbf{0})$, where $5 \leqslant i \leqslant 7$, identified with the subset of which it is a characteristic vector. Since M_{23} acts 4-transitively, and thus 2-transitively, on M, we may assume that $a, b \in \alpha$. Suppose three points $p_1, p_2, p_3 \in \alpha \setminus \{a, b\}$ form a triangle in $\mathrm{PG}_2(4)$. Then, by [13, Theorem 6.6B], there exists a hyperoval $\Delta \in \mathcal{H}_1$ such that $p_1, p_2, p_3 \in \Delta$, and $\Delta \cup \{a, b\} \in C$. It follows that $|\alpha \cap (\Delta \cup \{a, b\})| \geqslant 5$, which implies that

$$d(\alpha, \Delta \cup \{a, b\}) = |\alpha| + |\Delta \cup \{a, b\}| - 2|\alpha \cap (\Delta \cup \{a, b\})| \leqslant i - 2,$$

contradicting $\alpha \in C_i$. Thus, all points in $\alpha \setminus \{a, b\}$ lie on a single line of $\mathrm{PG}_2(4)$. Now $(\mathrm{M}_{23})_{a,b} = \mathrm{PSL}_3(4)$ acts transitively on the set of j-subsets, where $j = 3, 4, 5$, of collinear points in $\mathrm{PG}_2(4)$. Thus M_{23} acts transitively on $C_i \cap \Gamma_i(\mathbf{0})$ for $i = 5, 6, 7$, completing the proof. $\qquad \square$

The main purpose of next result is to help in determining the complete submodule structure of the permutation module over $\mathbb{F}_2$ for the action of $\mathrm{PSL}_3(4)$ on $\mathrm{PG}_2(4)$. However, we include it in this section as it is required to complete the proof of Lemma 5.4. Note that, as mentioned in Remark 3.5, the result below contradicts [13, Theorem 6.5B (iv)], which erroneously claims that $\mathrm{P\Gamma L}_3(4)/\mathrm{PSL}_3(4) \cong \mathbb{Z}_6$.

Lemma 5.3. *Let Δ_1 and Δ_2 be hyperovals of $\mathrm{PG}_2(4)$ such that Δ_1 and Δ_2 intersect in a triangle. Then the symmetric difference $\Delta_1 + \Delta_2 = \Delta_3$ is a hyperoval of $\mathrm{PG}_2(4)$ and each of Δ_1, Δ_2 and Δ_3 lie in different orbits under $\mathrm{PSL}_3(4)$. Moreover, the action induced by $\mathrm{P\Gamma L}_3(4)$ on the set $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ of $\mathrm{PSL}_3(4)$-orbits of hyperovals is that of S_3 on three points.*

Proof. Let ω be a generator of the multiplicative group $\mathbb{F}_4^\times$, so that ω satisfies $\omega^2 + \omega + 1 = 0$, and identify M with the set $\{(x, y, z) \mid x, y, z \in \mathbb{F}_4, (x, y, z) \neq (0, 0, 0)\}$ with the understanding that two triples (x_1, x_2, x_3) and (y_1, y_2, y_3) represent the same point of M whenever there exists some $c \in \mathbb{F}_4$ such that $x_i = cy_i$ for all $i = 1, 2, 3$. Moreover, let

$$\Delta_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1), (1, \omega, \omega^2), (1, \omega^2, \omega)\} \quad \text{and}$$
$$\Delta_2 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, \omega), (1, \omega, 1), (\omega, 1, 1)\},$$

so that
$$\Delta_3 = \{(1, 1, 1), (1, \omega, \omega^2), (1, \omega^2, \omega), (1, 1, \omega), (1, \omega, 1), (\omega, 1, 1)\}.$$

Then Δ_1 is indeed a hyperoval since it has the standard form

$$\{(1, t, t^2) \mid t \in \mathbb{F}_4\} \cup \{(0, 1, 0), (0, 0, 1)\},$$

made up of a conic and its nucleus (see, for instance, [27]). Also, Δ_2 is a hyperoval since it is the image of Δ_1 under the diagonal matrix with non-zero entries $1, 1, \omega$, corresponding to an element of $\mathrm{PGL}_3(4) \setminus \mathrm{PSL}_3(4)$. To see that Δ_3 is a hyperoval, consider the quadrangle

$$\Xi = \{(1, 1, 1), (1, 1, \omega), (1, \omega, 1), (\omega, 1, 1)\}.$$

The line through $(1, 1, 1), (1, 1, \omega)$ meets the line through $(1, \omega, 1), (\omega, 1, 1)$ at the point $(1, 1, 0)$, the line through $(1, 1, 1), (1, \omega, 1)$ meets the line through $(1, 1, \omega), (\omega, 1, 1)$ at the point $(1, 0, 1)$, and the line through $(1, 1, 1), (\omega, 1, 1)$ meets the line through $(1, 1, \omega), (1, \omega, 1)$ at the point $(0, 1, 1)$. These three points, $(1, 1, 0), (1, 0, 1), (0, 1, 1)$ all lie on the same line, and the remaining two points on this line, namely $(1, \omega, \omega^2)$ and $(1, \omega^2, \omega)$ are the only points of $\mathrm{PG}_2(4)$ not on one of the six lines through a pair of points of Ξ, and thus are the remaining points on the (unique) hyperoval containing Ξ. Thus Δ_3 is a hyperoval. Now, Δ_1, Δ_2 and Δ_3 all pairwise intersect in a triangle, and hence, by [13, Theorem 6.6B], are all in different $\mathrm{PSL}_3(4)$-orbits.

Let
$$\Delta_4 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, \omega^2), (1, \omega^2, 1), (\omega^2, 1, 1)\}.$$

Then, again by [13, Theorem 6.6B], $\Delta_1, \Delta_2, \Delta_4$ are in different $\mathrm{PSL}_3(4)$-orbits. The diagonal matrix with non-zero entries $1, 1, \omega$ then maps

$$\Delta_1 \mapsto \Delta_2 \mapsto \Delta_4 \mapsto \Delta_1.$$

The Frobenius automorphism fixes Δ_1 and interchanges Δ_2 and Δ_4, completing the proof. $\square$

We are now in a position to prove the last result of this section.

Lemma 5.4. *Suppose $m = 22$ and Hypothesis 2 holds with $\mathrm{soc}(X/K) \cong \mathrm{M}_{22}$. Then C is equivalent to one of following completely transitive codes in $H(22, 2)$.*

- *The $[22, 12, 6; 3]$ punctured binary Golay code $\mathcal{G}_{22}$.*

- *The $[22, 11, 6; 7]$ even weight subcode $\mathcal{E}_{22}$.*

- *The $[22, 11, 7; 6]$ shortened binary Golay code $\mathcal{S}_{22}$.*

Proof. Since M_{22} has index 2 in $\mathrm{Aut}(\mathrm{M}_{22})$ we have, by part 4 of Proposition 2.7 and part 3 of Lemma 2.10, that $C = C_{\mathsf{max}}$. The possibilities for C_{max} are given in Lemma 5.1. First, $\mathcal{G}_{22}^{\perp}$ has 11 non-zero weights, and thus C has external distance 11, but covering radius 7. By [36, Theorem 4.1], the covering radius of a completely regular code must equal its external distance, therefore $\mathcal{G}_{22}^{\perp}$ is not completely regular, and hence not completely transitive. Thus C is $\mathcal{G}_{22}$, $\mathcal{E}_{22}$ or $\mathcal{S}_{22}$.

As each of the codes $\mathcal{G}_{22}$, $\mathcal{E}_{22}$ or $\mathcal{S}_{22}$ are linear, Proposition 2.7 implies that $X = T_C \rtimes X_{\mathbf{0}}$, and in each case $T_C \leqslant X$ acts transitively on C. Thus, to show complete transitivity, it suffices to show that $X_{\mathbf{0}}$ acts transitively on $C_i \cap \Gamma_i(\mathbf{0})$ for $i = 1, \ldots, \rho$. Moreover, $\mathrm{M}_{22} \leqslant X_{\mathbf{0}}^M$ acts 3-transitively on M, and thus $X_{\mathbf{0}}$ acts transitively on $\Gamma_i(\mathbf{0})$, for each $i = 1, 2, 3$. Since $\mathcal{G}_{22}$ has covering radius $\rho = 3$, $\mathcal{G}_{22}$ is completely transitive. Recall that M here is the set of all points of $\mathrm{PG}_2(4)$ with the extra point a adjoined, as in Example 3.4.

Let $C = \mathcal{S}_{22}$. Then C contains the codeword $\Delta \cup \{a\}$ for each hyperoval $\Delta \in \mathcal{H}_1 \cup \mathcal{H}_3$. Consider a vertex $\alpha \in C_i \cap \Gamma_i(\mathbf{0})$, where $i = 4, 5$ or 6, identified with the corresponding subset of M. Since M_{22} acts transitively on M, we may assume that $a \in \alpha$. Suppose three points $p_1, p_2, p_3 \in \alpha \setminus \{a\}$ form a triangle in $\mathrm{PG}_2(4)$. Then, by [13, Theorem 6.6B], there exists a hyperoval $\Delta \in \mathcal{H}_1$ such that $p_1, p_2, p_3 \in \Delta$. It follows that $|\alpha \cap (\Delta \cup \{a\})| \geqslant 4$, which implies that

$$d(\alpha, \Delta \cup \{a\}) = |\alpha| + |\Delta \cup \{a\}| - 2|\alpha \cap (\Delta \cup \{a\})| \leqslant i - 1,$$

contradicting $\alpha \in C_i$. Thus, all points in $\alpha \setminus \{a\}$ lie on a single line of $\mathrm{PG}_2(4)$. Now $(\mathrm{M}_{22})_a = \mathrm{PSL}_3(4)$ acts transitively on the set of j-subsets, where $j = 3, 4, 5$, of collinear points in $\mathrm{PG}_2(4)$. Thus M_{22} acts transitively on $C_i \cap \Gamma_i(\mathbf{0})$ for $i = 4, 5, 6$, and C is completely transitive.

Let $C = \mathcal{E}_{22}$. Then, C contains the codewords $\ell \cup \{a\}$ for each line ℓ of $\mathrm{PG}_2(4)$ as well as the codewords Δ for each hyperoval $\Delta \in \mathcal{H}_2$. Note that here $X_{\mathbf{0}}^M \cong \mathrm{M}_{22} : 2 = \mathrm{Aut}(\mathrm{M}_{22})$ and $\mathrm{P\Sigma L}_3(4)$ is isomorphic to the stabiliser of a inside $X_{\mathbf{0}}^M$. Consider a vertex $\alpha \in C_i \cap \Gamma_i(\mathbf{0})$, where $i = 4, 5, 6$ or 7, identified with the corresponding subset of M. Since $X_{\mathbf{0}}$ acts transitively on M, we may assume that $a \in \alpha$. Suppose three distinct points $p_1, p_2, p_3 \in \alpha \setminus \{a\}$ lie on a line ℓ in $\mathrm{PG}_2(4)$. It follows that $|\alpha \cap (\ell \cup \{a\})| \geqslant 4$, which implies that

$$d(\alpha, \ell \cup \{a\}) = |\alpha| + |\ell \cup \{a\}| - 2|\alpha \cap (\ell \cup \{a\})| \leqslant i - 2,$$

contradicting $\alpha \in C_i$. Thus, no 3 points in $\alpha \setminus \{a\}$ lie on a single line of $\mathrm{PG}_2(4)$. It follows that $\alpha \setminus \{a\}$ is contained in some hyperoval Δ in $\mathcal{H}_1, \mathcal{H}_2$ or $\mathcal{H}_3$. If $i = 4$ then the group $X_{\mathbf{0},a} \cong \mathrm{P\Sigma L}_3(4)$ acts transitively on triangles, so that $X_{\mathbf{0}}$ acts transitively on $C_4 \cap \Gamma_4(\mathbf{0})$. Suppose $i = 5, 6$ or 7. If $\alpha \setminus \{a\} \subseteq \Delta \in \mathcal{H}_2$ then $d(\alpha, \Delta) \leqslant 3$, contradicting $\alpha \in C_i$. Hence, $\Delta \in \mathcal{H}_1$ or $\mathcal{H}_3$. By Lemma 5.3, there is an element of $X_{\mathbf{0},a}$ that interchanges $\mathcal{H}_1$ and $\mathcal{H}_3$ (and also interchanges b and c). Moreover, by [13, Exercise 6.5.13], the stabiliser of Δ inside this copy of $\mathrm{P\Sigma L}_3(4)$ acts as the symmetric group S_6 on the points of Δ. Thus, we have that $X_{\mathbf{0}}$ acts transitively on $C_i \cap \Gamma_i(\mathbf{0})$ for each $i = 5, 6, 7$. Hence C is completely transitive. $\square$

6 $\mathrm{PSL}_3(4)$

In this section we shall consider codes satisfying Hypothesis 2, where M is the point-set of the projective plane $\mathrm{PG}_2(4)$ and where $\mathrm{soc}(X/K) \cong \mathrm{PSL}_3(4)$. See Example 3.3 for the constructions of the codes arising in this section.

Lemma 6.1. *Assume Hypothesis 2 holds, $m = 21$ and $\mathrm{soc}(X/K) \cong \mathrm{PSL}_3(4)$. Then C_{max} contains the code $\mathcal{P}$, which has parameters $[21, 9, 8; 7]$, and is contained in the code $\mathcal{P}^{\perp}$, which has parameters $[21, 12, 5; 3]$.*

Proof. By [29, Section 8], the unique minimal submodule of the permutation module over $\mathbb{F}_2$ for the action of $\mathrm{PSL}_3(4)$ on $\mathrm{PG}_2(4)$, which we identify here with the vertex set of $H(21, 2)$, is $\mathcal{P}$, that is, the code generated by the set of all complements of lines in $\mathrm{PG}_2(4)$. Now, for any two distinct lines ℓ_1, ℓ_2 the sum of their complements is $(\ell_1 + M) + (\ell_2 + M) = \ell_1 + \ell_2$. Thus, it follows that $\mathcal{P}$ contains the code generated by the symmetric difference $\ell_1 + \ell_2$ of all pairs of lines ℓ_1, ℓ_2 of $\mathrm{PG}_2(4)$. Moreover, if $\ell_1, \ell_2, \ell_3, \ell_4$ are pairwise distinct lines in $\mathrm{PG}_2(4)$ all through a common point p, then $(\ell_1 + \ell_2) + (\ell_3 + \ell_4) = \ell_5 + M$, where ℓ_5 is the remaining line through p. As the complement of any line may be formed in this way, $\mathcal{P}$ is in fact generated by the set of all symmetric differences $\ell_1 + \ell_2$ of pairs ℓ_1, ℓ_2 of lines of $\mathrm{PG}_2(4)$. It follows that $\mathcal{P}$ is generated by only those blocks appearing under case 8 of Example 3.4 and therefore $\mathcal{P}$ is the shortened code of the $[22, 10, 8; 7]$ code $\mathcal{G}_{22}^{\perp}$ (see Lemma 5.1), and thus has parameters $[21, 9, 8; 7]$. Since Hypothesis 2 assumes that C_{max} has codimension at least 2 in $V\Gamma \cong \mathbb{F}_2^m$, it follows that C_{max} is contained in $\mathcal{P}^{\perp}$, which is the punctured code of the $[22, 12, 6; 3]$ code $\mathcal{G}_{22}$ (again, see Lemma 5.1), and so $\mathcal{P}^{\perp}$ has parameters $[21, 12, 5; 3]$. $\square$

The following lemma demonstrates an important relationship between the cosets of $\mathcal{P}$ in $\mathcal{P}^{\perp}$. Note that the sum in the conclusion should be interpreted in terms of characteristic vectors, and is equivalent to the symmetric difference of the subsets of points.

Lemma 6.2. *Let ℓ be a line of $\mathrm{PG}_2(4)$ and, for each $i = 1, 2, 3$, let Δ_i be a hyperoval in $\mathcal{H}_i$ and Φ_i be a Fano subplane in $\mathcal{F}_i$, as in Example 3.3. Then ℓ, Δ_i and Φ_i may be chosen so that $\Delta_i = \ell + \Phi_i$.*

Proof. Let ω be a generator of the multiplicative group $\mathbb{F}_4^{\times}$, so that ω satisfies $\omega^2 + \omega + 1 = 0$, and identify M with the set $\{(x, y, z) \mid x, y, z \in \mathbb{F}_4, (x, y, z) \neq (0, 0, 0)\}$ with the understanding that two triples (x_1, x_2, x_3) and (y_1, y_2, y_3) represent the same point of M whenever there exists some $c \in \mathbb{F}_4$ such that $x_i = cy_i$ for all $i = 1, 2, 3$. Let

$$\ell = \{(1, 0, 0), (0, 1, 0), (1, 1, 0), (1, \omega, 0), (1, \omega^2, 0)\},$$

and let

$$\Phi_1 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}.$$

Consider the subset

$$\ell + \Phi_1 = \{(1, \omega, 0), (1, \omega^2, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 1)\},$$

of M. Now the four points $(0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 1)$ form a quadrangle Ξ, and considering each pair of points in the quadrangle gives six lines. These six lines contain all the points of M except for $(1, \omega, 0)$ and $(1, \omega^2, 0)$, and hence $\ell + \Phi_1$ is the unique hyperoval containing the quadrangle Ξ. Moreover, since Φ_1 and $\ell + \Phi_1$ intersect precisely in Ξ, it follows that $\ell + \Phi_1$ is a hyperoval in $\mathcal{H}_1$ (see Example 3.3). Thus we may let $\Delta_1 = \ell + \Phi_1$. Let τ be the permutation on M defined by $\tau : (x, y, z) \mapsto (x, y, z\omega)$. Since τ corresponds to a linear transformation with determinant ω, τ corresponds to an element of $\mathrm{PGL}_3(4) \setminus \mathrm{PSL}_3(4)$. Now, by Lemma 5.3, the quotient group $\mathrm{P\Gamma L}_3(4)/\mathrm{PSL}_3(4)$ acts as S_3 on the set $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$, from which it follows that τ induces a 3-cycle on each of the sets $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ and $\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3\}$. Note that τ also fixes the line ℓ. Hence, we may choose $\Phi_2 = \Phi_1^{\tau}$, $\Phi_3 = \Phi_1^{\tau^2}$, $\Delta_2 = \Delta_1^{\tau}$ and $\Delta_3 = \Delta_1^{\tau^2}$ so that $\Delta_i = \ell + \Phi_i$ for each $i = 1, 2, 3$. This completes the proof. $\square$
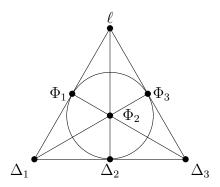
Figure 1: Arrangement of the non-trivial cosets of $\mathcal{P}$ in $\mathcal{P}^\perp$, labelled by their representatives as in the proof of Lemma 6.4.

We now determine the structure of the submodule lattice, which in turn allows us to determine all the possibilities for $C_{\max}$.

Lemma 6.3. *The submodule lattice of the permutation module over $\mathbb{F}_2$ for the action of $\mathrm{PSL}_3(4)$ on $\mathrm{PG}_2(4)$, identified with $V\Gamma$, is as in Figure 2.*

Proof. By Lemma 6.1, $\mathcal{P}$ and $\mathcal{P}^\perp$ are, respectively, the unique minimal and unique maximal non-trivial submodules. For $i = 1, 2, 3$, let ℓ, Δ_i, Φ_i be as in Example 3.3, chosen as in Lemma 6.2, so that $\Delta_i = \ell + \Phi_i$. We claim that the Fano plane in Figure 1 represents the configuration of the non-trivial cosets of $\mathcal{P}$ in $\mathcal{P}^\perp$, where a vertex with label α represents the coset $\mathcal{P} + \alpha$. First, $\mathcal{P}$ has codimension 3 in $\mathcal{P}^\perp$, and hence the non-trivial cosets of $\mathcal{P}$ may be represented by the points of a Fano plane. Also, $\mathcal{P}$ has minimum distance 8 and so, for $\alpha = \ell, \Delta_i, \Phi_i$ with $i = 1, 2, 3$, we see that $\mathcal{P}$ does not contain α, and hence $\mathcal{P} + \alpha \neq \mathcal{P}$. Note that since $\mathcal{P}^\perp$ has covering radius 3, $\mathcal{P}^\perp$ does indeed contain each of ℓ, Δ_i, Φ_i for $i = 1, 2, 3$. Now, $\Delta_i = \ell + \Phi_i$ for each $i = 1, 2, 3$. Moreover, since each coset of $\mathcal{P}$ also has minimum distance 8 we deduce that each of $\mathcal{P} + \ell$, $\mathcal{P} + \Delta_i$ and $\mathcal{P} + \Phi_i$ are pairwise distinct, for $i = 1, 2, 3$, giving the three lines through ℓ. By Lemma 5.3, $\mathcal{P} + \Delta_1$, $\mathcal{P} + \Delta_2$ and $\mathcal{P} + \Delta_3$ are distinct cosets and lie on a common line, which then forces the remainder of the configuration. Thus our claim holds.

Each module in the submodule lattice, as in Figure 2, may then be deduced from Figure 1: the points of the Fano plane correspond to the codes $\mathcal{L}$, $\langle\mathcal{P}, \Delta_i\rangle$, $\langle\mathcal{P}, \Phi_i\rangle$, for $i = 1, 2, 3$ and the lines of the Fano plane correspond to the codes $\langle\mathcal{P}, \Delta_1, \Delta_2, \Delta_3\rangle$, $\langle\mathcal{P}, \ell, \Delta_i, \Phi_i\rangle$ and $\langle\mathcal{P}, \Delta_i, \Phi_{i+1}, \Phi_{i+2}\rangle$, for $i = 1, 2, 3$ and with subscripts modulo 3. Now, all codewords of $\mathcal{P}$ have even weight, and hence $\mathcal{P}$ does not contain the repetition code $\langle\mathbf{1}\rangle$, whilst set M is the sum of the five lines through a given point, and hence $\mathcal{L}$ does contain $\langle\mathbf{1}\rangle$. The remaining inclusions in Figure 2 follow from the incidences of Figure 1, completing the proof. $\square$

Lemma 6.4. *Suppose $m = 21$ and Hypothesis 2 holds with $\mathrm{soc}(X/K) \cong \mathrm{PSL}_3(4)$. Then C_{max} is equivalent to one of following codes:*

- *The $[21, 12, 5; 3]$ code $\mathcal{P}^\perp$ with $X_\mathbf{0} \cong \mathrm{P\Gamma L}_3(4)$.*

- *The $[21, 11, 5; 6]$ code $\langle\mathcal{L}, \Delta_1\rangle = \langle\mathcal{P}, \ell, \Delta_1, \Phi_1\rangle$ with $X_\mathbf{0} \cong \mathrm{P\Sigma L}_3(4)$.*

- *The $[21, 11, 6; 5]$ code $\langle\mathcal{P}, \Phi_1\rangle^\perp = \langle\mathcal{P}, \Delta_1, \Phi_2, \Phi_3\rangle$ with $X_\mathbf{0} \cong \mathrm{P\Sigma L}_3(4)$.*

- *The $[21, 11, 6; 5]$ code $\mathcal{L}^\perp = \langle\mathcal{P}, \Delta_1, \Delta_2, \Delta_3\rangle$ with $X_\mathbf{0} \cong \mathrm{P\Gamma L}_3(4)$.*

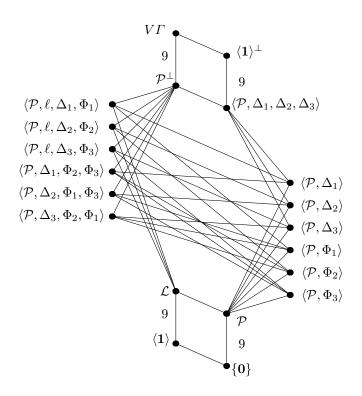Figure 2: The submodule lattice of the permutation module over $\mathbb{F}_2$ for the action of $\mathrm{PSL}_3(4)$ on the points of $\mathrm{PG}_2(4)$. Each vertex is labelled with the corresponding code; each edge indicates the codimension of the corresponding inclusion, with an empty label denoting codimension 1. For definitions of the codes see Example 3.3 and for further information see Lemma 6.3.

- *The $[21, 10, 5; 6]$ code $\mathcal{L} = \langle \mathcal{P}, \ell \rangle$ with $X_\mathbf{0} \cong \mathrm{P\Gamma L}_3(4)$.*

- *The $[21, 10, 6; 7]$ code $\langle \mathcal{P}, \Delta_1 \rangle = \langle \mathcal{L}, \Delta_1 \rangle^\perp$ with $X_\mathbf{0} \cong \mathrm{P\Sigma L}_3(4)$.*

- *The $[21, 10, 7; 6]$ code $\langle \mathcal{P}, \Phi_1 \rangle$ with $X_\mathbf{0} \cong \mathrm{P\Sigma L}_3(4)$.*

- *The $[21, 9, 8; 7]$ code $\mathcal{P}$ with $X_\mathbf{0} \cong \mathrm{P\Gamma L}_3(4)$.*

Proof. Let $C = C_{\mathrm{max}}$. Since $2 \leqslant \dim(C) \leqslant m - 2$, the possibilities for C are determined by the non-trivial submodules as in Lemma 6.3 and Figure 2, from which the dimension of each code and equivalences between codes can be deduced. The equality $\langle \mathcal{L}, \Delta_1 \rangle = \langle \mathcal{P}, \ell, \Delta_1, \Phi_1 \rangle$ follows from the fact that $\mathcal{L} = \langle \mathcal{P}, \ell \rangle$ and from Lemma 6.2, and Figure 2 implies the equality $\mathcal{L}^\perp = \langle \mathcal{P}, \Delta_1, \Delta_2, \Delta_3 \rangle$. In each case $X_\mathbf{0}$ can be deduced by considering the Fano plane in Figure 1 and the fact that $\mathrm{PSL}_3(4)$ fixes it point-wise while, by Lemma 5.3, $\mathrm{P\Gamma L}_3(4)$ fixes $\mathcal{P} + \ell$ and induces S_3 on $\{\mathcal{P} + \Delta_i \mid i = 1, 2, 3\}$. Indeed, the codes containing precisely one orbit of hyperovals, or precisely one or two orbits of Fano subplanes, satisfy $X_\mathbf{0} \cong \mathrm{P\Sigma L}_3(4)$ whilst the remaining codes satisfy $X_\mathbf{0} \cong \mathrm{P\Gamma L}_3(4)$. Note that there are three copies of $\mathrm{P\Sigma L}_3(4)$ inside $\mathrm{P\Gamma L}_3(4)$ and these corresdond to the stabilisers of each of the three lines through ℓ in Figure 1. The equalities

$$\langle \mathcal{P}, \Phi_1 \rangle^\perp = \langle \mathcal{P}, \Delta_1, \Phi_2, \Phi_3 \rangle \quad \text{and} \quad \langle \mathcal{P}, \Delta_1 \rangle^\perp = \langle \mathcal{L}, \Delta_1 \rangle,$$

among the four codes that are invariant under the same copy of $\mathrm{P\Sigma L}_3(4)$, can be seen from the fact that $\langle \mathcal{P}, \Delta_1 \rangle$ is contained in three pairwise inequivalent codes of dimension 11 but $\langle \mathcal{P}, \Phi_1 \rangle$ is contained in a pair of equivalent codes of dimension 11, whilst $\langle \mathcal{L}, \Delta_1 \rangle$ contains three inequivalent codes of dimension 10 but $\langle \mathcal{P}, \Delta_1, \Phi_2, \Phi_3 \rangle$ contains a pair of equivalent codes of dimension 10.

Now, the set of all lines of $\mathrm{PG}_2(4)$ form the set of weight 5 codewords of $\mathcal{P}^\perp$; the set of all hyperovals of $\mathrm{PG}_2(4)$ form the set of weight 6 codewords of $\mathcal{P}^\perp$; and the set of all Fano subplanes of $\mathrm{PG}_2(4)$ form the set of weight 7 codewords of $\mathcal{P}^\perp$. Thus, the codes $\langle \mathcal{L}, \Delta_1 \rangle$ and $\mathcal{L}$ both have minimum distance 5; the codes $\langle \mathcal{P}, \Phi_1 \rangle^\perp$, $\mathcal{L}^\perp$ and $\langle \mathcal{P}, \Delta_1 \rangle$ have minimum distance 6; and $\langle \mathcal{P}, \Phi_1 \rangle$ has minimum distance 7. The covering radii were determined via computation in the GAP package GUAVA [11]. $\qquad\square$

Lemma 6.5. *Suppose $m = 21$ and Hypothesis 2 holds with $\mathrm{soc}(X/K) \cong \mathrm{PSL}_3(4)$. If $C = C_{max}$, then C is equivalent to one of the following codes in $H(21, 2)$, each of which is completely transitive:*

- *The $[21, 12, 5; 3]$ code $\mathcal{P}^\perp$,*

- *The $[21, 11, 5; 6]$ code $\langle \mathcal{L}, \Delta_1 \rangle$, or,*

- *The $[21, 10, 5; 6]$ code $\mathcal{L}$.*

Proof. The possibilities for C_{max} are given in Lemma 6.4. Using the GAP package GUAVA [11] to compare the covering radius of a code with the number of non-zero weights of its dual, which for a completely regular code must be equal by [8, Theorem 2.15 (iii)], allows many of the possibilities to be eliminated. The covering radius of $\langle \mathcal{P}, \Phi_1 \rangle^\perp$ is 5, but $\langle \mathcal{P}, \Phi_1 \rangle$ has 6 non-zero weights, so $\langle \mathcal{P}, \Phi_1 \rangle^\perp$ is not completely regular, and hence not completely transitive. The covering radius of $\mathcal{L}^\perp$ is 5, but $\mathcal{L}$ has 7 non-zero weights, so $\mathcal{L}^\perp$ is not completely transitive. The covering radius of $\langle \mathcal{P}, \Delta_1 \rangle$ is 7, but $\langle \mathcal{P}, \Delta_1 \rangle^\perp = \langle \mathcal{L}, \Delta_1 \rangle$ has 13 non-zero weights, so $\langle \mathcal{P}, \Delta_1 \rangle$ is not completely transitive. The covering radius of $\langle \mathcal{P}, \Phi_1 \rangle$ is 6, but $\langle \mathcal{P}, \Phi_1 \rangle^\perp$ has 9 non-zero weights, so $\langle \mathcal{P}, \Phi_1 \rangle$ is not completely transitive. The covering radius of $\mathcal{P}$ is 7, but $\mathcal{P}^\perp$ has 13 non-zero weights, so $\mathcal{P}$ is not completely transitive.

Now, $\mathcal{P}^\perp$ has covering radius 3 and minimum distance 5, so $\Gamma_3(\mathbf{0}) \cap \left(\mathcal{P}^\perp\right)_3$ consists of the set of all triangles of $\mathrm{PG}_2(4)$. As $\mathrm{PSL}_3(4)$ acts transitively on triangles we have that $\mathcal{P}^\perp$ is completely transitive.

The code $\langle \mathcal{L}, \Delta_1 \rangle$ has covering radius 6 and contains all lines of $\mathrm{PG}_2(4)$ as well as one $\mathrm{PSL}_3(4)$-orbit of hyperovals and one $\mathrm{PSL}_3(4)$-orbit of Fano subplanes. Thus, $\Gamma_3(\mathbf{0}) \cap \langle \mathcal{L}, \Delta_1 \rangle_3$ again consists of the set of all triangles of $\mathrm{PG}_2(4)$, on which $\mathrm{PSL}_3(4)$ acts transitively. Since each quadrangle of $\mathrm{PG}_2(4)$ is contained in a unique hyperoval and a unique Fano subplane, $\Gamma_4(\mathbf{0}) \cap \langle \mathcal{L}, \Delta_1 \rangle_4$ consists of two $\mathrm{PSL}_3(4)$-orbits of quadrangles. However, $X_{\mathbf{0}} \cong \mathrm{P\Sigma L}_3(4)$ acts transitively on $\{\mathcal{H}_2, \mathcal{H}_3\}$. Hence $\mathrm{P\Sigma L}_3(4)$ acts transitively on $\Gamma_4(\mathbf{0}) \cap \langle \mathcal{L}, \Delta_1 \rangle_4$. Similarly, two $\mathrm{PSL}_3(4)$-orbits of ovals of $\mathrm{PG}_2(4)$ make up $\Gamma_5(\mathbf{0}) \cap \langle \mathcal{L}, \Delta_1 \rangle_5$. Again, these orbits are interchanged under the field automorphism of $\mathbb{F}_4$, and hence $\mathrm{P\Sigma L}_3(4)$ acts transitively on $\Gamma_5(\mathbf{0}) \cap \langle \mathcal{L}, \Delta_1 \rangle_5$. Moreover, the remaining two $\mathrm{PSL}_3(4)$-orbits of hyperovals of $\mathrm{PG}_2(4)$ make up $\Gamma_6(\mathbf{0}) \cap \langle \mathcal{L}, \Delta_1 \rangle_6$, these orbits are again interchanged under the field automorphism of $\mathbb{F}_4$, so that $\mathrm{P\Sigma L}_3(4)$ acts transitively on $\Gamma_6(\mathbf{0}) \cap \langle \mathcal{L}, \Delta_1 \rangle_6$. It follows that $\mathcal{P}(\Delta)^\perp$ is completely transitive.

Finally, $\mathcal{L}$ has covering radius 6 and contains all lines of $\mathrm{PG}_2(4)$. Now $\mathrm{P\Gamma L}_3(4)$ acts transitively on the quadrangles, ovals and hyperovals of $\mathrm{PG}_2(4)$, respectively, which make up $\Gamma_4(\mathbf{0}) \cap \mathcal{L}_4$, $\Gamma_5(\mathbf{0}) \cap \mathcal{L}_5$ and $\Gamma_6(\mathbf{0}) \cap \mathcal{L}_6$, respectively. Since $\mathcal{L}$ is also invariant under $\mathrm{P\Gamma L}_3(4)$ acting on M it follows that $\mathcal{L}$ is completely transitive. $\qquad\square$

In the remainder of this section we consider $C \neq C_{max}$, introducing two new non-linear codes. One of these codes is 2-neighbour-transitive, but not completely transitive, while the other is completely transitive.
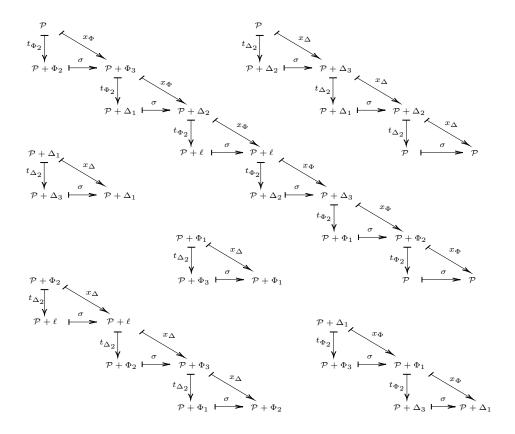
Figure 3: Images of the cosets of $\mathcal{P}$ in $\mathcal{P}^\perp$ under the induced action of $x_\Delta = t_{\Delta_2}\sigma$ and $x_\Phi = t_{\Phi_2}\sigma$ as in the proofs of Lemmas 6.6 and 6.7.

Lemma 6.6. *Let* $C = \langle\mathcal{P}, \Delta_2\rangle \cup \langle\mathcal{P}, \Delta_3\rangle$. *Then* C *has parameters* $(21, 2^9 \cdot 3, 6; 7)$ *and is* 2-*neighbour-transitive, but not* 3-*neighbour-transitive, having automorphism group* $\mathrm{Aut}(C) \cong T_\mathcal{P} \rtimes \mathrm{P\Gamma L}_3(4)$ *with* $\mathrm{Aut}(C)_\mathbf{0} \cong \mathrm{P\Sigma L}_3(4)$.

Proof. First, $T_\mathcal{P}$ has three orbits on C, namely $\mathcal{P}$, $\mathcal{P} + \Delta_2$ and $\mathcal{P} + \Delta_3$. By Lemma 5.3, we know that $\mathrm{P\Gamma L}_3(4)$, as a subgroup of the top group L, induces an action of S_3 on $\{\mathcal{P} + \Delta_i \mid i = 1, 2, 3\}$ with kernel $\mathrm{PSL}_3(4)$. Hence, for each i, the stabiliser of $\mathcal{P} + \Delta_i$ is some copy of $\mathrm{P\Sigma L}_3(4)$, with this copy of $\mathrm{P\Sigma L}_3(4)$ still acting transitively on $\{\mathcal{P} + \Delta_j \mid j \in \{1, 2, 3\}; j \neq i\}$. Thus, $\mathrm{Aut}(C)_\mathbf{0} \cong \mathrm{P\Sigma L}_3(4)$. Let $x_\Delta = t_{\Delta_2}\sigma \in \mathrm{Aut}(\Gamma)$, where σ is an element of $\mathrm{PGL}_3(4) \leqslant L$ inducing the permutation:

$$\mathcal{P} + \Delta_1 \mapsto \mathcal{P} + \Delta_2 \mapsto \mathcal{P} + \Delta_3 \mapsto \mathcal{P} + \Delta_1.$$

Then, applying Lemma 5.3 to find the images under t_{Δ_2}, Figure 3 gives the orbit of $\mathcal{P}$ under $\langle x_\Delta \rangle$. It follows that $\mathrm{Aut}(C) = T_\mathcal{P} \rtimes \langle \mathrm{P\Sigma L}_3(4), x_\Delta \rangle$, and also that $\mathrm{Aut}(C)$ acts transitively on C. Since $\mathrm{Aut}(C)_\mathbf{0}$ is 2-transitive on M it follows that C is 2-neighbour-transitive. Moreover, no element of $\mathrm{Aut}(C)$ maps three collinear points of $\mathrm{PG}_2(4)$ to a triangle, so that $\mathrm{Aut}(C)$ is not transitive on C_3, and thus C is not 3-neighbour-transitive. Finally, since $\Phi_1 \in \mathcal{F}_1$ satisfies $|\Delta \cap \Phi_1| \geqslant 4$ for all $\Delta \in \mathcal{H}_2 \cup \mathcal{H}_3 \subseteq C$, and hence $d(\Delta, \Phi_1) \geqslant 7$, we have that $\Phi_1 \in C + 7$ and C has covering radius 7. $\square$

Lemma 6.7. *Suppose* $m = 21$ *and Hypothesis 2 holds with* $\mathrm{soc}(X/K) \cong \mathrm{PSL}_3(4)$. *If* $C \neq C_{max}$ *then, up to equivalence,* $C = \langle\mathcal{L}, \Delta_2\rangle \cup \langle\mathcal{L}, \Delta_3\rangle$ *with parameters* $(21, 2^{10} \cdot 3, 5; 6)$, *which is indeed completely transitive with* $\mathrm{Aut}(C) \cong T_\mathcal{L} \rtimes \mathrm{P\Gamma L}_3(4)$ *and* $\mathrm{Aut}(C)_\mathbf{0} \cong \mathrm{P\Sigma L}_3(4)$.

Proof. Suppose $C \neq C_{\mathsf{max}}$ and C is completely transitive. Note that $\mathrm{PSL}_3(4)$ has index 6 in $\mathrm{P\Gamma L}_3(4)$, so that, by Proposition 2.7, $|C|/|C_{\mathsf{max}}|$ divides 6. Furthermore, by Lemma 2.10, we have that either $|C| = 3|C_{\mathsf{max}}|$ or $|C| = 6|C_{\mathsf{max}}|$. It follows that X induces an orbit of either size 3 or 6 on some set of cosets of $\mathcal{P}$. By Lemma 5.3, we know that $\mathrm{P\Gamma L}_3(4)$, as a subgroup of the top group L, induces an action of S_3 on the sets $\{\mathcal{P} + \Delta_i \mid i = 1, 2, 3\}$ and $\{\mathcal{P} + \Phi_i \mid i = 1, 2, 3\}$ with kernel $\mathrm{PSL}_3(4)$. Indeed, for each i, the stabiliser of $\mathcal{P} + \Delta_i$, which is equal to the stabiliser of $\mathcal{P} + \Phi_i$, is some copy of $\mathrm{P\Sigma L}_3(4)$ that acts transitively on the sets $\{\mathcal{P} + \Delta_j \mid j \in \{1, 2, 3\}; j \neq i\}$ and $\{\mathcal{P} + \Phi_j \mid j \in \{1, 2, 3\}; j \neq i\}$.

Let σ be an element of $\mathrm{PGL}_3(4) \leqslant L$ inducing the permutation:

$$\mathcal{P} + \Delta_1 \mapsto \mathcal{P} + \Delta_2 \mapsto \mathcal{P} + \Delta_3 \mapsto \mathcal{P} + \Delta_1.$$

Moreover, let t_{Δ_2} be the translation by Δ_2, t_{Φ_2} be the translation by Φ_2, $x_\Delta = t_{\Delta_2}\sigma$ and $x_\Phi = t_{\Phi_2}\sigma$. Figure 3 gives the images of each of the cosets of $\mathcal{P}$ under x_Δ and x_Φ. By the above considerations on $|C|/|C_{\mathsf{max}}|$ we see that the relevant orbits are the two orbits of length 3 under $\langle x_\Delta \rangle$ and the orbit of length 6 under $\langle x_\Phi \rangle$. Individually, the length 3 orbits correspond to the code $\langle \mathcal{P}, \Delta_2 \rangle \cup \langle \mathcal{P}, \Delta_3 \rangle$ and its translate by ℓ. These codes are not completely transitive by Lemma 6.6. However, their union is the same code given by the orbit of length 6, namely $\langle \mathcal{L}, \Delta_2 \rangle \cup \langle \mathcal{L}, \Delta_3 \rangle$. Note that the actions of $\langle x_\Delta \rangle$ and $\langle x_\Phi \rangle$ cover, up to equivalence, all possibilities. Hence it remains to show that $\langle \mathcal{L}, \Delta_2 \rangle \cup \langle \mathcal{L}, \Delta_3 \rangle$ is completely transitive.

Let $C = \langle \mathcal{L}, \Delta_2 \rangle \cup \langle \mathcal{L}, \Delta_3 \rangle$. Then $\mathrm{Aut}(C) = T_{\mathcal{L}} \rtimes \langle \mathrm{P\Sigma L}_3(4), x_\Delta \rangle$ and, since $T_{\mathcal{L}}$ acts transitively on $\mathcal{L}$ and $\langle x_\Delta \rangle$ acts transitively on $\{\mathcal{L}, \mathcal{L} + \Delta_2, \mathcal{L} + \Delta_3\}$, it follows that $\mathrm{Aut}(C)$ acts transitively on C. Now $\langle \mathcal{L}, \Delta_2 \rangle$ has covering radius 6, and hence C has covering radius $\rho \leqslant 6$. For each hyperoval $\Delta \in \mathcal{H}_2 \cup \mathcal{H}_3$ we have that $d(\Delta, \Delta_1) \geqslant 6$, and hence $\rho = 6$. Since C contains all lines of $\mathrm{PG}_2(4)$, we have that $\Gamma_3(\mathbf{0}) \cap C_3$ is the set of triangles, $\Gamma_4(\mathbf{0}) \cap C_4$ is one $\mathrm{PSL}_3(4)$-orbit of quadrangles, $\Gamma_5(\mathbf{0}) \cap C_5$ is one $\mathrm{PSL}_3(4)$-orbit of ovals, and $\Gamma_6(\mathbf{0}) \cap C_6$ is one $\mathrm{PSL}_3(4)$-orbit of hyperovals of $\mathrm{PG}_2(4)$. It follows that $\mathrm{Aut}(C)_{\mathbf{0}}$ acts transitively on $\Gamma_i(\mathbf{0}) \cap C_i$ for each $i = 1, \ldots, 6$. Thus C is completely transitive. $\qquad\square$

7 Higman–Sims and Conway Groups

Again, assume that Hypothesis 2 holds. We note that both of the groups considered here are equal to their automorphism group. Proposition 2.7 then implies that $C = C_{\mathsf{max}}$.

First, consider the case that $\mathrm{soc}(X/K)$ is the Higman–Sims group HS. From the proof of [29, Theorem 8.1] we have that the $\mathbb{F}_2$-permutation module for the 2-transitive action of HS is uniserial with structure $1\backslash 20\backslash 1\backslash 132\backslash 1\backslash 20\backslash 1$. Codes invariant under this action are constructed in [33, Section 5], and verified by our own GAP [15] computations. The codes constructed are a $[176, 21, 56]$ code, a $[176, 22, 50]$ code, a $[176, 154, 6]$ code and a $[176, 155, 6]$ code, each invariant under HS. Since the permutation module is uniserial, these codes, along with the repetition code and its dual, give all proper submodules invariant under HS. By Lemma 2.6, it follows that C is either the $[176, 154, 6]$ code or the $[176, 155, 6]$ code. By [8, Theorem 2.12 (ii)], the weight 6 codewords of C form a 3-$(176, 6, \lambda)$ design, for some integer λ, and hence $|\Gamma_6(\mathbf{0}) \cap C|$ is divisible by

$$\frac{\binom{176}{3}}{\binom{6}{3}} = 44660.$$

However, the number of weight 6 codewords of the $[176, 154, 6]$ code and the $[176, 155, 6]$ code are 92400 and 129360 respectively. Thus, these codes are not completely regular and hence also not completely transitive.

Turning to consider $\mathrm{soc}(X/K)$ isomorphic to the Conway group Co_3, the proof of [29, Theorem 8.1] says that the $\mathbb{F}_2$-permutation module for the 2-transitive action of Co_3 is uniserial with structure $1\backslash 22\backslash 230\backslash 22\backslash 1$. A $[276, 23, 100]$ code invariant under Co_3 is given in [25] with its dual, by computations in GAP [15], being a $[276, 253, 6]$ code. Since the permutation module is uniserial, these codes, along with the repetition code and its dual, give all proper submodules invariant under Co_3. By Lemma 2.6, it follows that C is the $[276, 253, 6]$ code. By [8, Theorem 2.12 (ii)], the weight 6 codewords of C form a 3-$(276, 6, \lambda)$ design, for some integer λ, and hence $|\Gamma_6(\mathbf{0}) \cap C|$ is divisible by

$$\frac{\binom{276}{3}}{\binom{6}{3}} = 173305.$$

However, the number of weight 6 codewords of the $[276, 253, 6]$ code is 708400. Thus, this code is not completely regular and hence also not completely transitive.

In summary:

Lemma 7.1. *There are no codes satisfying Hypothesis 2 with $\mathrm{soc}(X/K) = \mathrm{HS}$ or Co_3.*

8 $\mathrm{PSL}_2(r)$

Suppose that Hypothesis 2 holds, and that $\mathrm{soc}(X/K) \cong \mathrm{PSL}_2(r)$ where $m = r + 1 \geqslant 24$ with $r = p^t$ for some prime p and integer t. It follows from [30, Theorem 1] and Lemma 2.6 that either:

1. $r \equiv 3 \pmod 4$, $\delta \leqslant 8$ and $X_{\mathbf{0}}^M$ is 3-homogeneous, but not 4-homogeneous; or,

2. $r \equiv 1 \pmod 4$, $\delta \leqslant 6$ and $X_{\mathbf{0}}^M$ is 2-homogeneous, but not 3-homogeneous.

Lemma 8.1. *Suppose $m = r + 1$, where $r = p^t$ with p prime, and Hypothesis 2 holds with $\mathrm{soc}(X/K) \cong \mathrm{PSL}_2(r)$. Then C_{max} has dimension $\frac{r+1}{2}$ and $t \leqslant 4$.*

Proof. By [29, Lemma 5.4], we have that C_{max} is one of two equivalent codes U_1 and U_2, where U_1 and U_2 are interchanged under the action $\mathrm{PGL}_2(r)$. Furthermore, U_1 and U_2 are self-dual when $r \equiv 3 \pmod 4$, and $U_1^{\perp} = U_2$ when $r \equiv 1 \pmod 4$. Either way, we have that U_1 and U_2 have the same dimensions and minimum distances. Hence C_{max} has dimension $\frac{r+1}{2}$.

Suppose $t \geqslant 5$. Now [26, Lemma 4.4] gives the bound $\delta_{\mathrm{max}} \geqslant \sqrt{r} + 1$. Since r is odd, and hence $p \geqslant 3$, we have that $r \geqslant 243$ and $\delta_{\mathrm{max}} \geqslant 17$. We claim that

$$|C|/|C_{\mathrm{max}}| < \frac{m(m-1)}{\delta(\delta-1)},$$

in which case we may apply Lemma 2.8, which implies that $\delta_{\mathrm{max}} \leqslant 2\delta$. Since $\delta \leqslant 8$, but $\delta_{\mathrm{max}} \geqslant 17$, we then have a contradiction, and hence $t \leqslant 4$. It remains to prove the claim.

First, $\mathrm{PSL}_2(r)$ has index $2t$ in $\mathrm{P\Gamma L}_2(r)$. Thus, by Proposition 2.7, $|C|/|C_{\mathrm{max}}| = |X^M : X_{\mathbf{0}}^M|$ is at most $2t$. Now, for all $t \geqslant 5$ the inequality $2t < 3^t(3^t + 1)/56$ holds. Using this, and since $\delta \leqslant 8$ and $m - 1 = r = p^t \geqslant 3^t$, we obtain:

$$\frac{|C|}{|C_{\mathrm{max}}|} \leqslant 2t < \frac{3^t(3^t + 1)}{7 \cdot 8} \leqslant \frac{m(m-1)}{\delta(\delta-1)},$$

thus proving the claim. $\qquad\square$

Lemma 8.2. *Suppose $m = r + 1$ and Hypothesis 2 holds with $\mathrm{soc}(X/K) \cong \mathrm{PSL}_2(r)$. Then $r \in \{23, 25, 31, 41\}$.*

Proof. By Lemma 8.1, we have that the dimension of C_{max} is $\frac{r+1}{2}$. Proposition 2.7 part 3 implies that $X/T_{C_{max}} \cong X^M$ and hence $|X| = |T_{C_{max}}||X^M|$. Since $X^M \leqslant \mathrm{P\Gamma L}_2(r)$, we have that

$$|X| = |T_{C_{max}}||X^M| \leqslant 2^{(r+1)/2}r(r+1)(r-1)t.$$

It then follows from Lemma 2.9 that

$$2^{r+1} \leqslant (m+1)|X| \leqslant (r+2)2^{(r+1)/2}r(r+1)(r-1)t. \tag{8.1}$$

Moreover, Lemma 8.1 implies that $t \leqslant 4$, so that the above becomes

$$2^{(r+1)/2} \leqslant 4r(r+2)(r+1)(r-1).$$

This holds for $r = 23, 25, 31, 41$ and 47. Note that $t = 1$ when $r = 47$, in which case the inequality (8.1) becomes $2^{(r+1)/2} \leqslant (r+2)r(r+1)(r-1)$, which no longer holds. $\square$

Lemma 8.3. *Suppose $m = r + 1$ and Hypothesis 2 holds with $\mathrm{soc}(X/K) \cong \mathrm{PSL}_2(r)$. Then $C = C_{max}$.*

Proof. Suppose $C \neq C_{max}$. By Lemma 8.2, we have that $t \leqslant 2$. Since $\mathrm{soc}(X_0/K) = \mathrm{soc}(X/K) \cong \mathrm{PSL}_2(r)$, we have that $\mathrm{PSL}_2(r) \leqslant X_0^M$ and $X^M \leqslant \mathrm{P\Gamma L}_2(r)$. Therefore $|X^M : X_0^M|$ divides 4 and hence, by Proposition 2.7 part 5, we have that $|C|/|C_{max}|$ divides 4. Lemma 2.10 part 3 then implies that $|C|/|C_{max}| \neq 2$ and hence that $|C|/|C_{max}| = 4$. Applying Lemma 2.10 part 3 once more, we have that C_{max} has codimension 2 or 3 in the code $\langle C \rangle$ spanned by the codewords of C, which must be also be completely transitive. Hence $\langle C \rangle$ has dimension $(r+5)/2$ or $(r+7)/2$. By [29, Lemma 5.4], the only submodules of dimension greater than $(r+1)/2$ that are invariant under $\mathrm{PSL}_2(r)$ are $\langle \mathbf{1} \rangle^{\perp}$ and $V\Gamma$. This implies that $(r+5)/2$ or $(r+7)/2$ is equal to r or $r+1$, in which case $r = 3, 5$ or 7. However this contradicts Lemma 8.2. $\square$

Lemma 8.4. *There are no completely transitive codes satisfying Hypothesis 2 with $\mathrm{soc}(X/K) \cong \mathrm{PSL}_2(r)$.*

Proof. By Lemma 8.2, we have that $r \in \{23, 25, 31, 41\}$, by Lemma 8.3, we have that $C = C_{max}$ and, by [29, Lemma 5.4], there are only two possibilities for C_{max}, which are in fact equivalent codes. First, by [32, p. 482], if $r = 23$ then C is the extended Golay code $\mathcal{G}_{24}$, in which case $X/K \cong \mathrm{M}_{24} \neq \mathrm{PSL}_2(23)$.

Suppose $r = 25$. Then a calculation in GAP involving the package GUAVA shows that both of the two possibilities for C_{max} here have external distance 9 but covering radius 5, and thus, by [36, Theorem 4.1], are not completely regular.

Let $r = 31$. The two codes that arise here have minimum distance 8, and hence, by [8, Theorem 2.12 (ii)], the weight 8 codewords form a 4-design. This implies that the number of weight 8 codewords is a multiple of 3596. However, there are only 620 such codewords in each code.

Finally, let $r = 41$. Then a calculation in GAP using the GUAVA package shows that C_{max} has minimum distance 10, and is thus not completely transitive. $\square$

9 $\mathrm{PSU}_3(r)$

Suppose Hypothesis 2 holds and $\mathrm{soc}(X/K) \cong \mathrm{PSU}_3(r)$ with r an odd prime power.

Lemma 9.1. *Suppose $r = p^t$ is an odd prime power, $m = r^3 + 1$ and Hypothesis 2 holds with* $\operatorname{soc}(X/K) \cong \operatorname{PSU}_3(r)$. *Then $r = 3$ or 5.*

Proof. By the remark following Theorem 7.3 in [29], any code invariant under X_0 that has dimension at least 2 must have dimension at least $r^2 - r + 1$. By duality, it follows that the dimension of $C_{\max}$ is at most

$$m - (r^2 - r + 1) = r^3 - r^2 + r.$$

Now $X/K \leqslant \operatorname{P\Gamma U}_3(r)$. Hence, applying Lemma 2.9,

$$2^{r^3+1} \leqslant (m+1)|X| \leqslant (r^3 + 2)2^{r^3-r^2+r}r^3(r^2 - 1)(r^3 + 1)t.$$

Thus, since $t \leqslant r$ holds, r satisfies

$$2^{r^2-r+1} \leqslant (r^3 + 2)r^4(r^2 - 1)(r^3 + 1).$$

This holds for an odd prime power r only when $r = 3$ or 5. $\qquad\square$

Lemma 9.2. *There are no codes satisfying Hypothesis 2 with $\operatorname{soc}(X/K) \cong \operatorname{PSU}_3(r)$.*

Proof. By [30], X_0^M is 2-transitive but not 3-homogeneous, and therefore Lemma 2.6 implies that $\delta = 5$ or 6. By Lemma 9.1, we have that $r = 3$ or 5. We first consider the case $r = 3$. Suppose $\delta = 5$. Then, by Lemma 2.11, the weight 5 codewords form a 2-$(28, 5, \lambda)$ design, for some integer $\lambda \leqslant (m-2)/3$. That is, $\lambda \leqslant 8$. Now, the number of blocks of such a design is:

$$\frac{28 \cdot 27}{5 \cdot 4}\lambda.$$

Since this number must be an integer, we have that 5 divides λ, and hence $\lambda = 5$. Thus, there must be 189 weight 5 codewords. However, an analysis in GAP [15] shows that the smallest orbit on 5-sets for $\operatorname{PSU}_3(3)$ in its 2-transitive action has size 1512.

Suppose $\delta = 6$. Then, by Lemma 2.11, the weight 6 codewords form a 3-$(28, 6, \lambda)$ design, for some integer $\lambda \leqslant (m-3)/3$. That is, $\lambda \leqslant 8$. Now, the number of blocks of such a design is:

$$\frac{28 \cdot 27 \cdot 26}{6 \cdot 5 \cdot 4}\lambda.$$

Since this number must be an integer, we have that 5 divides λ, and hence $\lambda = 5$. Thus, there must be 819 weight 6 codewords. However, an analysis in GAP [15] shows that, for $\operatorname{PSU}_3(3)$ in its 2-transitive action, the orbits on 6-subsets having length at most 819 either have size 756 or 504.

Let $r = 5$ and k be the dimension of $C_{\max}$. It follows from [29, Theorem 7.2] $C_{\max}$ is either contained in the code U generated by the unital (a 2-$(126, 6, 1)$ design; see [13, Section 7.7] for more information), which has dimension 105, or its dual $U^\perp$, which has dimension 21. By Lemma 2.9, we have that

$$(m + 1)2^k|\operatorname{PGU}_3(5)| \geqslant 2^m.$$

This implies that $k \geqslant 104$, so that $k = 104$ or 105, and hence that either $C_{\max} = U$ or $C_{\max}$ has codimension 1 in U. By [29, Theorem 7.2] there is no such code of codimension 1 in U, so $C_{\max} = U$. Since $\operatorname{PSU}_3(5)$ has index 3 in $\operatorname{PGU}_3(5)$, part 5 of Proposition 2.7 and Lemma 2.10 imply that $C = C_{\max}$ or $C_{\max}$ has codimension 2 in $\langle C \rangle$. By [29, Theorem 7.2], there is no linear code invariant under X_0 having dimension 107, and hence $C = C_{\max} = U$. Constructing U in the GAP package GUAVA, we see that it has minimum distance 6 and 21525 codewords of

weight 6. By Lemma 2.11, the weight 6 codewords form a 3-$(126, 6, \lambda)$ design for some integer λ. Now, the number of blocks of such a design is:

$$\frac{126 \cdot 125 \cdot 124}{6 \cdot 5 \cdot 4}\lambda = 21 \cdot 25 \cdot 31\lambda.$$

This gives a contradiction, since 21525 is not divisible by 31. $\qquad\square$

10 $\mathrm{Ree}(r)$

Suppose Hypothesis 2 holds and that $\mathrm{soc}(X/K) \cong \mathrm{Ree}(r)$ with $r = 3^{2t+1}$ and $m = r^3 + 1$.

Lemma 10.1. *Suppose r is an odd power of 3, that $m = r^3 + 1$ and Hypothesis 2 holds with $\mathrm{soc}(X/K) \cong \mathrm{Ree}(r)$. Then $r = 3$.*

Proof. By [29, Lemma 7.4], any code invariant under X_0 that has dimension at least 2 must have dimension at least $r^2 - r + 1$. By duality, it follows that the dimension of C_{max} is at most

$$m - (r^2 - r + 1) = r^3 - r^2 + r.$$

Since $X/K \leqslant \mathrm{Ree}(r)$, we have that $|X/K|$ divides $r^3(r^3 + 1)(r - 1)(2t + 1)$. Hence, applying Lemma 2.9, we have,

$$2^{r^3+1} \leqslant (m + 1)|X| \leqslant (r^3 + 2)2^{r^3 - r^2 + r}r^3(r^3 + 1)(r - 1)(2t + 1).$$

Thus, since $(2t + 1) \leqslant r$ holds, r satisfies

$$2^{r^2-r+1} \leqslant (r^3 + 2)r^4(r^3 + 1)(r - 1).$$

This holds for $r = 3^{2t+1}$ only when $r = 3$. $\qquad\square$

Lemma 10.2. *There are no codes satisfying Hypothesis 2 with $\mathrm{soc}(X/K) \cong \mathrm{Ree}(r)$.*

Proof. By [30], X_0^M is 2-transitive but not 3-homogeneous. Thus, Lemma 2.6 implies that $\delta = 5$ or 6. By Lemma 10.1, we have that the only possibility for is $r = 3$. Suppose $\delta = 5$. Then, by Lemma 2.11, the weight 5 codewords form a 2-$(28, 5, \lambda)$ design, for some integer $\lambda \leqslant (m-2)/3$. That is, $\lambda \leqslant 8$. Now, the number of blocks of such a design is:

$$\frac{28 \cdot 27}{5 \cdot 4}\lambda.$$

Since this number must be an integer, we have that 5 divides λ, and hence $\lambda = 5$. Thus, there must be 189 weight 5 codewords. However, an analysis in GAP [15] shows that the smallest orbit on 5-sets for $\mathrm{Ree}(3)$ in its 2-transitive action has size 756.

Suppose $\delta = 6$. Then, by Lemma 2.11, the weight 6 codewords form a 3-$(28, 6, \lambda)$ design, for some integer $\lambda \leqslant (m-3)/3$. That is, $\lambda \leqslant 8$. Now, the number of blocks of such a design is:

$$\frac{28 \cdot 27 \cdot 26}{6 \cdot 5 \cdot 4}\lambda.$$

Since this number must be an integer, we have that 5 divides λ, and hence $\lambda = 5$. Thus, there must be 819 weight 6 codewords. However, an analysis in GAP [15] shows that, for $\mathrm{Ree}(3)$ in its 2-transitive action, each orbit on 6-subsets has even length. This completes the proof. $\qquad\square$

11 Proof of main result

We now prove Theorem 1.3. Suppose that C is a non-trivial completely transitive code in $H(m, 2)$ with minimum distance $\delta \geqslant 5$. Let $C_{\max}$ be the maximal linear subcode of C, let $X = \mathrm{Aut}(C)$ and let $X_{\max} = X_{C_{\max}}$. It then follows from Proposition 2.7 part 3 that $K = X \cap B$ is equal to $X_{\max} \cap B = T_{C_{\max}}$. Furthermore, suppose that $\mathrm{soc}(X_{\max}/K) \cong G$, where G is one of the groups appearing in Table 1. Note that $\delta \geqslant 5$ implies $C_2 \neq \emptyset$ from which it follows that C is 2-neighbour-transitive and that Theorem 1.5 applies.

Suppose that $C_{\max}$ is contained in the binary repetition code $\langle \mathbf{1} \rangle$. It follows that C comes under part 1 or 2 of Theorem 1.5. Now, by [16, Lemma 4.2], the even weight subcode of the punctured Hadamard code is not completely transitive. Moreover, by [18, Corollary 5], both the Hadamard code of length 12 and its punctured code are completely transitive. This gives lines 1 and 2 of Table 2. Hence, we may assume that $C_{\max}$ has dimension at least 2 for the remainder of the proof, which implies that C is as in part 3 of Theorem 1.5. In particular, this means that M_{11} and M_{12} do not need to be considered in the remainder of the proof as they do not appear in [26, Table 1].

Now, Theorem 1.5 implies that the repetition code in $H(m, 2)$ is the unique linear code of dimension 1 stabilised by X_0. It follows that the dual $\langle \mathbf{1} \rangle^{\perp}$ of the binary repetition code is the only linear code of codimension 1 stabilised by X_0. Since $\langle \mathbf{1} \rangle^{\perp}$ has minimum distance 2 it follows that $C \neq \langle \mathbf{1} \rangle^{\perp}$. Thus $2 \leqslant \dim(C_{\max}) \leqslant m - 2$ and Hypothesis 1 holds. Suppose that $\mathrm{soc}(X/K) \neq \mathrm{soc}(X_{\max}/K)$. Then, by Propositions 4.2 and 4.3, we have that $m = 15$, $\mathrm{soc}(X_{\max}/K) \cong \mathrm{A}_7$ and $C = \mathcal{NR}$, as in line 3 of Table 2. Moreover, by [22, Theorem 1.1], $\mathcal{NR}$ is completely transitive. Note that Proposition 4.3 means that we do not need to consider $\mathrm{soc}(X_{\max}/K) = \mathrm{A}_7$ in the remainder of the proof.

We now have that $\mathrm{soc}(X/K) = \mathrm{soc}(X_{\max}/K)$ and so Hypothesis 2 holds. Suppose that $\mathrm{soc}(X/K) = \mathrm{PSL}_3(4)$. Then, by Lemmas 6.5 and 6.7, we have that C is one of the codes appearing on lines 4–7 of Table 2, each of which is completely transitive. Let $\mathrm{soc}(X/K) = \mathrm{M}_m$ with $m = 22, 23$ or 24. Then, by Lemmas 5.2 and 5.4, it follows that one of the lines 8–13 holds, and that the codes appearing there are indeed completely transitive. By Lemmas 7.1, 8.4, 9.2 and 10.2, no codes satisfying Hypothesis 2 exist with $\mathrm{soc}(X/K) = \mathrm{HS}, \mathrm{Co}_3, \mathrm{PSL}_2(r), \mathrm{PSU}_3(r)$ or $\mathrm{Ree}(r)$. This completes the proof.

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