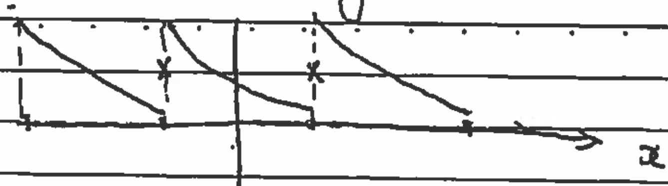
66

Solution to Assignment 3.

3.2.2. (b)



sketch of Fourier series of f(x)

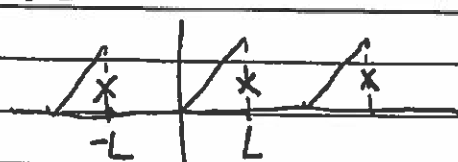
4

$$a_0 = \frac{1}{2L}\int_{-L}^{L} e^{-x}\,dx = \frac{e^{L}-e^{-L}}{2L}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\cos\frac{n\pi x}{L}\,dx = \frac{\cos n\pi\,(e^{L}-e^{-L})}{L\left(1+\left(\frac{n\pi}{L}\right)^2\right)} = \frac{2\cos n\pi\,\sinh(L)}{L\left(1+\left(\frac{n\pi}{L}\right)^2\right)}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} e^{-x}\sin\frac{n\pi x}{L}\,dx = \frac{\frac{n\pi}{L}\cos n\pi\,(e^{L}-e^{-L})}{L\left(1+\left(\frac{n\pi}{L}\right)^2\right)}$$

(d).



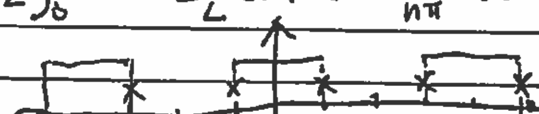
4

$$a_0 = \frac{1}{2L}\int_0^L x\,dx = \frac{L}{4}$$

$$a_n = \frac{1}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx = \frac{L}{(n\pi)^2}\left(\cos n\pi - 1\right)$$

$$b_n = \frac{1}{L}\int_0^L x\sin\frac{n\pi x}{L}\,dx = -\frac{L}{n\pi}\cos n\pi.$$

(e).



4

$$a_0 = \frac{1}{2L}\int_{-\frac{L}{2}}^{L/2} 1\,dx = \frac{1}{2}.$$

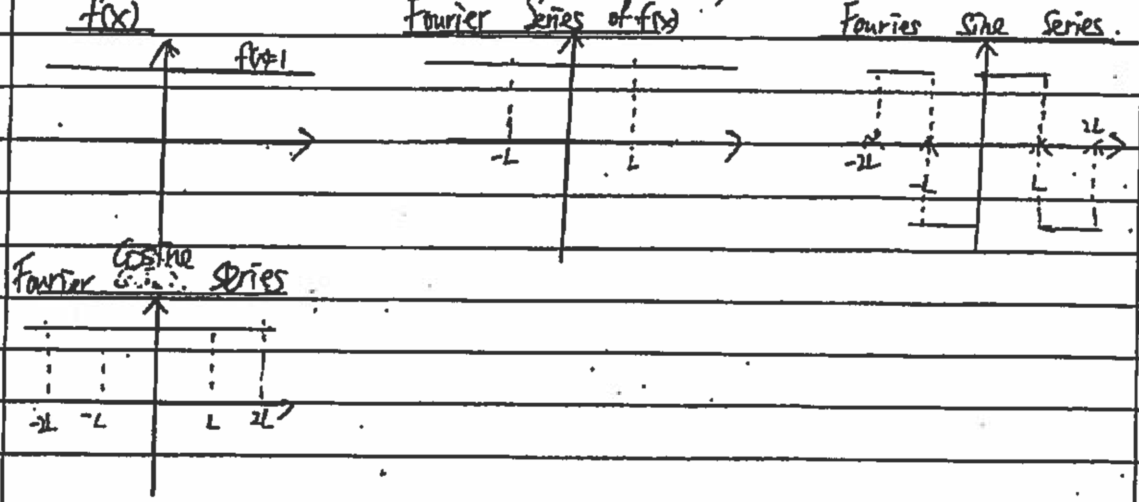
$$a_n = \frac{1}{L}\int_{-\frac{L}{2}}^{L/2}\cos\frac{n\pi x}{L}\,dx = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}\sin\frac{n\pi x}{L}\,dx = 0$$

Subject 科目：________ Name 姓名：________ Class 班別：______ No 學號：______

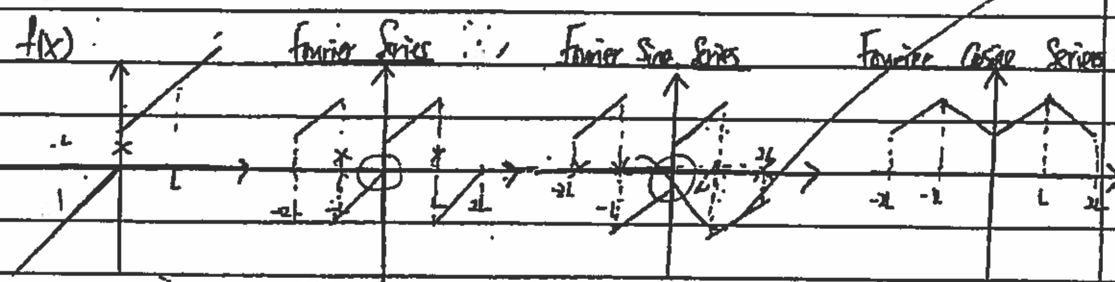
3.3.1 a

4



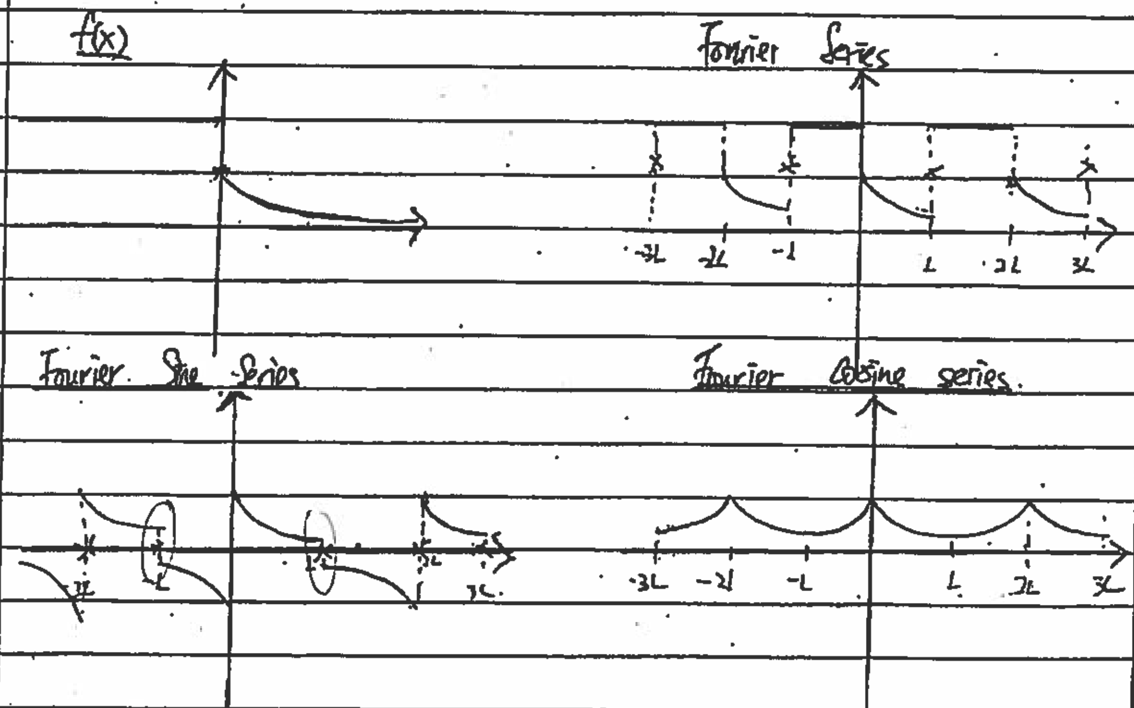
3.3.1 c

4

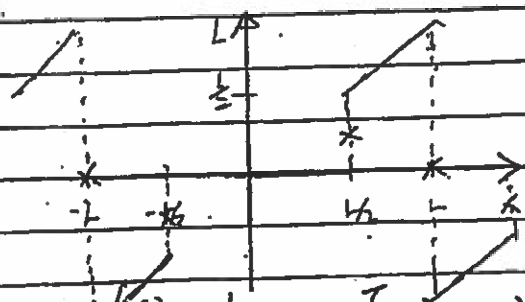


3.3.1 e

4



3.3.2 (a)



$f(x)$ has Fourier sine series

$\therefore\ B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$= \frac{2}{L}\int_0^{L/2} 0\cdot\sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/2}^{L} x\sin\frac{n\pi x}{L}\,dx$

$= -\frac{2}{L}\left(\frac{L}{n\pi}\right)\left[\int_{L/2}^{L} x\, d\cos\frac{n\pi x}{L}\right]$

$= -\frac{2}{n\pi}\left\{\left[x\cos\frac{n\pi x}{L}\right]_{L/2}^{L} - \int_{L/2}^{L}\cos\frac{n\pi x}{L}\,dx\right\}$

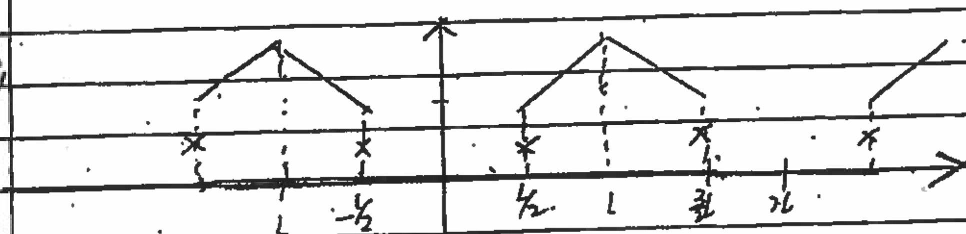
$= -\frac{2}{n\pi}\left[L\cos n\pi - \frac{L}{2}\cos\frac{n\pi}{2}\right] + \frac{2}{n\pi}\int_{L/2}^{L}\cos\frac{n\pi x}{L}\,dx$

$= -\frac{2L}{n\pi}\cos n\pi + \frac{L}{n\pi}\cos\frac{n\pi}{2} + \frac{2L}{(n\pi)^2}\left[\sin\frac{n\pi x}{L}\right]_{L/2}^{L}$

$= -\frac{2L}{n\pi}\cos n\pi + \frac{L}{n\pi}\cos\frac{n\pi}{2} - \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$ //

4

3.3.5 (a)



4

$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\int_{L/2}^{L} x\,dx$

$= \frac{1}{L}\left[\frac{x^2}{2}\right]_{L/2}^{L}$

$= \frac{1}{L}\left[\frac{L^2}{2}\right] - \frac{1}{L}\left[\frac{L^2}{8}\right]$

$= \frac{3L}{8}$ //

$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_{L/2}^{L} x\cos\frac{n\pi x}{L}\,dx$

$= \frac{2}{n\pi}\left[x\sin\frac{n\pi x}{L}\right]_{L/2}^{L} - \frac{2}{n\pi}\int_{L/2}^{L}\sin\frac{n\pi}{L}x\,dx$

$= -\frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2L}{(n\pi)^2}\left[\cos\frac{n\pi}{L}x\right]_{L/2}^{L}$

$= -\frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2L}{(n\pi)^2}\cos n\pi - \frac{2L}{(n\pi)^2}\cos\frac{n\pi}{2}$ //

~~34~~

Solution to Assignment 3.

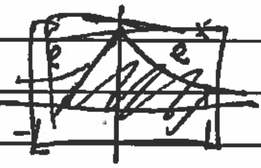
~~3.3.18. The solution can be found in the text book (omitted)~~

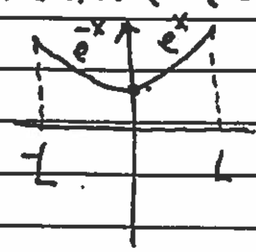
3.4.6. By the ~~the~~ theory of Fourier cosine series, e^x

5

can be evenly extended to the interval $(-L,0)$ so that

$$f(x)=\begin{cases} e^x, & x\in(0,L), \\ e^{-x}, & x\in(-L,0), \end{cases}$$





$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}$$

with $A_0 = \frac{1}{L}\int_0^L e^x\,dx$, $A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

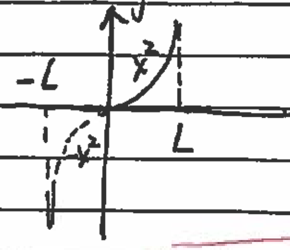
$$= \frac{2}{L}\int_0^L e^x \cos\frac{n\pi x}{L}\,dx.$$

The differentiation of $f(x)$ gives a function $f'(x)$ which is not continuous at least at $x=0$ except other points.

Therefore, the formula $e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin\frac{n\pi x}{L}$ can't be differentiable. In other word, the differentiation of this identy will result in a wrong formula!

3.5.1. $x^2 \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$.

odd extension of x^2 to have



(Fig.)

5 (a) from (3.5.6) we have. $\boxed{\frac{x^2}{2} \sim \frac{L}{2}x - \frac{4L^2}{\pi^3}\left(\sin\frac{\pi x}{L} + \frac{\sin\frac{3\pi x}{L}}{3^3} + \cdots\right)}$

from (3.2.11) and (3.3.12), we get

$$x \sim \sum_{n=1}^{\infty} \frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}$$

substituting this gives.

$$x^2 \sim L\cdot\sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L} - \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin\frac{(2n-1)\pi x}{L}}{(2n-1)^3}$$

$$\sim \sum_{n=1}^{\infty}\left(\frac{2L^2}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L} - \frac{8L^2}{\pi^3}\frac{\sin\frac{(2n-1)\pi x}{L}}{(2n-1)^3}\right).$$

2 (b). for $x \in (0, L)$, the above is an equality (see the fig)

(c). After integration, we have

5 $$x^3 \sim 3\sum_{n=1}^{\infty}\left(\frac{2L^3}{(n\pi)^2}(-1)^n\cos\frac{n\pi x}{L} + \frac{8L^3}{\pi^4}\frac{\cos\frac{(2n-1)\pi x}{L}}{(2n-1)^4}\right) + C$$

with $C = \frac{1}{L}\int_0^L x^3\,dx = \frac{L^3}{4}$.

4.4.3 (a) The term $-\beta\frac{\partial u}{\partial t}$ is called the resistence

(2) of the force of friction. It is proportional to the speed $\frac{\partial u}{\partial t}$, but with the minus sign meaning that the friction force is oppsite to movement direction.

(b). $u = h(t)\,\phi(x)$.

(5)
$$\begin{cases} \phi'' + \lambda\phi = 0, \quad \phi(0) = \phi(L) = 0 \\ \rho_0 h'' + \beta h' + \lambda T_0 h(t) = 0 \end{cases}$$

$$r = \frac{\beta}{-2\rho_0} \pm wi, \qquad w = \frac{\sqrt{\frac{4(n\pi)^2}{L}\rho_0 T_0 - \beta^2}}{2\rho_0}.$$

$$h = e^{-\frac{\beta}{2\rho_0}t}\left(c_1 \sin wt + c_2 \cos wt\right).$$

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\frac{\beta}{2\rho_0}t}\left(a_n \cos wt + b_n \sin wt\right)\sin\frac{n\pi x}{L}.$$

$u(x,0) = f(x)$: $\quad \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = f(x) \Rightarrow a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

$\frac{\partial u}{\partial t}(x,0) = g(x)$: $\quad \sum_{n=1}^{\infty} -\frac{\beta}{2\rho_0} a_n \sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty} w b_n \sin\frac{n\pi x}{L} = g(x)$

$$-\frac{\beta}{2\rho_0} f(x) + \sum_{n=1}^{\infty} w b_n \sin\frac{n\pi x}{L} = g(x)$$

$$w b_n = \frac{2}{L}\int_0^L \left(g(x) + \frac{\beta}{2\rho_0} f(x)\right)\sin\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{2}{Lw}\cdot\int_0^L \left(g(x) + \frac{\beta}{2\rho_0} f(x)\right)\sin\frac{n\pi x}{L}\,dx$$

⑤ 4.4.7 $u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi c}{L} t\right) + b_n \sin\left(\frac{n\pi c}{L} t\right) \right] \sin\frac{n\pi x}{L}$

$u(x,0) = f(x), \quad u_t(x,0) = g(x) = 0$

$g(x) = 0 \Rightarrow b_n = 0.$

$\underline{u(x,0) = f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}} \qquad 0 < x < L$

when x is in the whole interval $(-\infty, \infty)$.

$$\sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = F(x),$$

$F(x)$ is the odd periodic extension of $f(x)$.

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cdot \cos\left(\frac{n\pi c}{L} t\right) \cdot \sin\frac{n\pi x}{L}$$

$$= \sum_{n=1}^{\infty} a_n \cdot \frac{1}{2} \left(\sin\left(\frac{n\pi}{L}(x+ct)\right) + \sin\left(\frac{n\pi}{L}(x-ct)\right) \right)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}(x+ct)\right)$$

$$+ \frac{1}{2} \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}(x-ct)\right)$$

$$= \frac{1}{2} F(x+ct) + \frac{1}{2} F(x-ct).$$

$\square$

4.4.8. $u = \phi(x)h(t)$.

(5) $\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = \phi(L) = 0 \end{cases} \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2, \quad \phi(x) = \sin\frac{n\pi x}{L}$

$h'' + \lambda c^2 h = 0. \qquad h = c_1 \cos\left(\frac{n\pi c}{L}t\right) + c_2 \sin\left(\frac{n\pi c}{L}t\right)$

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi c}{L}t\right) + b_n \sin\left(\frac{n\pi c}{L}t\right)\right)\sin\frac{n\pi x}{L}.$$

$u(x,0) = f(x) = 0 \Rightarrow \quad a_n = 0$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi c}{L}t\right)\sin\frac{n\pi x}{L}.$$

$\frac{\partial u}{\partial t}(x,0) = g(x) \Rightarrow . \quad \sum_{n=1}^{\infty} b_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L} = g(x).$

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi c t}{L}\right)\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} b_n \frac{\cos\left(\frac{n\pi}{L}(x-ct)\right) - \cos\left(\frac{n\pi}{L}(x+ct)\right)}{2} \quad 0 < x < L$$

Given $G(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L}\sin\frac{n\pi x}{L}$. is the odd periodic extension of $g(x)$, $x \in \mathbb{R}$.

So. It is easy to work out

$$\frac{1}{2c}\int_{x-ct}^{x+ct} G(\bar{x})\,d\bar{x} = \sum_{n=1}^{\infty}\frac{1}{2c}\int_{x-ct}^{x+ct} B_n \frac{n\pi c}{L}\sin\frac{n\pi\bar{x}}{L}\,d\bar{x}$$

$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi c t}{L}\cdot\sin\frac{n\pi x}{L}$$

$$= u(x,t).$$