5.3.5. For the Sturm-Liouville eigenvalue problem,

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(L) = 0,$$

verify the following general properties:

(a) There is an infinite number of eigenvalues with a smallest but no largest.

(b) The nth eigenfunction has $n-1$ zeros.

(c) The eigenfunctions are complete and orthogonal.

(d) What does the Rayleigh quotient say concerning negative and zero eigenvalues?

5.3.6. Redo Exercise 5.3.5 for the Sturm-Liouville eigenvalue problem

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0 \quad \text{and} \quad \phi(L) = 0.$$

5.3.7. Which of statements 1–5 of the theorems of this section are valid for the following eigenvalue problem?

$$\begin{aligned} \tfrac{d^2\phi}{dx^2} + \lambda\phi &= 0 \quad \text{with} \\ \phi(-L) &= \phi(L) \\ \tfrac{d\phi}{dx}(-L) &= \tfrac{d\phi}{dx}(L). \end{aligned}$$

5.3.8. Show that $\lambda \geq 0$ for the eigenvalue problem

$$\frac{d^2\phi}{dx^2} + (\lambda - x^2)\phi = 0 \quad \text{with} \quad \frac{d\phi}{dx}(0) = 0, \quad \frac{d\phi}{dx}(1) = 0.$$

Is $\lambda = 0$ an eigenvalue?

5.3.9. Consider the eigenvalue problem

$$x^2\frac{d^2\phi}{dx^2} + x\frac{d\phi}{dx} + \lambda\phi = 0 \quad \text{with} \quad \phi(1) = 0, \quad \text{and} \quad \phi(b) = 0. \qquad (5.3.10)$$

(a) Show that multiplying by $1/x$ puts this in the Sturm-Liouville form. (This multiplicative factor is derived in Exercise 5.3.3.)

(b) Show that $\lambda \geq 0$.

*(c) Since (5.3.10) is an equidimensional equation, determine all positive eigenvalues. Is $\lambda = 0$ an eigenvalue? Show that there is an infinite number of eigenvalues with a smallest, but no largest.

(d) The eigenfunctions are orthogonal with what weight according to Sturm-Liouville theory? Verify the orthogonality using properties of integrals.

(e) Show that the nth eigenfunction has $n-1$ zeros.

5.3.10. Reconsider Exercise 5.3.9 with the boundary conditions

$$\frac{d\phi}{dx}(1) = 0 \quad \text{and} \quad \frac{d\phi}{dx}(b) = 0.$$

This approximation is not very good if $a_1 = 0$, in which case (5.4.14) should begin with the first nonzero term. However, often the initial temperature $f(x)$ is nonnegative (and not identically zero). In this case, we will show from (5.4.13) that $a_1 \neq 0$:

$$a_1 = \frac{\int_0^L f(x)\phi_1(x)c(x)\rho(x)\,dx}{\int_0^L \phi_1^2(x)c(x)\rho(x)\,dx}. \tag{5.4.15}$$

It follows that $a_1 \neq 0$, because $\phi_1(x)$ is the eigenfunction corresponding to the lowest eigenvalue and has no zeros; $\phi_1(x)$ is of one sign. Thus, if $f(x) > 0$ it follows that $a_1 \neq 0$, since $c(x)$ and $\rho(x)$ are positive physical functions. In order to sketch the solution for large fixed t, (5.4.14) shows that all that is needed is the first eigenfunction. At the very least, a numerical calculation of the first eigenfunction is easier than the computation of the first hundred.

For large time, the "shape" of the temperature distribution in space stays approximately the same in time. Its amplitude grows or decays in time depending on whether $\lambda_1 > 0$ or $\lambda_1 < 0$ (it would be constant in time if $\lambda_1 = 0$). Since this is a heat flow problem with no sources and with zero temperature at $x = 0$, we certainly expect the temperature to be exponentially decaying toward 0° (i.e., we expect that $\lambda_1 > 0$). Although the right end is insulated, heat energy should flow out the left end since there $u = 0$. We now prove mathematically that all $\lambda > 0$. Since $p(x) = K_0(x), q(x) = 0$, and $\sigma(x) = c(x)\rho(x)$, it follows from the Rayleigh quotient that

$$\lambda = \frac{\int_0^L K_0(x)(d\phi/dx)^2\,dx}{\int_0^L \phi^2 c(x)\rho(x)\,dx}, \tag{5.4.16}$$

where the boundary contribution to (5.4.16) vanished due to the specific homogeneous boundary conditions, (5.4.7) and (5.4.8). It immediately follows from (5.4.16) that all $\lambda \geq 0$, since the thermal coefficients are positive. Furthermore, $\lambda > 0$, since ϕ = constant is not an allowable eigenfunction [because $\phi(0) = 0$]. Thus, we have shown that $\lim_{t\to\infty} u(x,t) = 0$ for this example.

EXERCISES 5.4

5.4.1. Consider

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + \alpha u,$$

where c, ρ, K_0, α are functions of x, subject to

$$\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

Assume that the appropriate eigenfunctions are known.

(a) Show that the eigenvalues are positive if $\alpha < 0$ (see Sec. 5.2.1).

(b) Solve the initial value problem.

(c) Briefly discuss $\lim_{t\to\infty} u(x,t)$.

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*5.4.2. Consider

$$c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right),$$

where c, ρ, K_0 are functions of x, subject to

$$\begin{aligned} \tfrac{\partial u}{\partial x}(0,t) &= 0 \\ \tfrac{\partial u}{\partial x}(L,t) &= 0 \\ u(x,0) &= f(x). \end{aligned}$$

Assume that the appropriate eigenfunctions are known. Solve the initial value problem, briefly discussing $\lim_{t\to\infty} u(x,t)$.

*5.4.3. Solve

$$\frac{\partial u}{\partial t} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

with $u(r,0) = f(r), u(0,t)$ bounded, and $u(a,t) = 0$. You may assume that the corresponding eigenfunctions, denoted $\phi_n(r)$, are known and are complete. (*Hint:* See Sec. 5.2.2.)

5.4.4. Consider the following boundary value problem:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \text{ with } \frac{\partial u}{\partial x}(0,t) = 0 \text{ and } u(L,t) = 0.$$

Solve such that $u(x,0) = \sin \pi x/L$ (initial condition). (*Hint:* If necessary, use a table of integrals.)

5.4.5. Consider

$$\rho\frac{\partial u^2}{\partial t^2} = T_0\frac{\partial^2 u}{\partial x^2} + \alpha u,$$

where $\rho(x) > 0, \alpha(x) < 0$, and T_0 is constant, subject to

$$\begin{aligned} u(0,t) &= 0 \qquad u(x,0) = f(x) \\ u(L,t) &= 0 \qquad \tfrac{\partial u}{\partial t}(x,0) = g(x). \end{aligned}$$

Assume that the appropriate eigenfunctions are known. Solve the initial value problem.

*5.4.6. Consider the vibrations of a *nonuniform* string of mass density $\rho_0(x)$. Suppose that the left end at $x = 0$ is fixed and the right end obeys the elastic boundary condition: $\partial u/\partial x = -(k/T_0)u$ at $x = L$. Suppose that the string is *initially at rest* with a known initial position $f(x)$. Solve this initial value problem. (*Hints: Assume* that the appropriate eigenvalues and corresponding eigenfunctions are known. What differential equations with what boundary conditions do they satisfy? The eigenfunctions are orthogonal with what weighting function?)

are eigenvalue problems. In general, for a partial differential equation in N variables that completely separates. there will be N ordinary differential equations, $N-1$ of which are one-dimensional eigenvalue problems (to determine the $N-1$ separation constants). We have already shown this for $N=3$ (this section) and $N=2$.

EXERCISES 7.3

7.3.1. Consider the heat equation in a two-dimensional rectangular region $0<x<L, 0<y<H$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the initial condition

$$u(x,y,0) = f(x,y).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

*(a) $u(0,y,t)=0$, $u(L,y,t)=0$, $u(x,0,t)=0$, $u(x,H,t)=0$

(b) $\frac{\partial u}{\partial x}(0,y,t)=0$, $\frac{\partial u}{\partial x}(L,y,t)=0$, $\frac{\partial u}{\partial y}(x,0,t)=0$, $\frac{\partial u}{\partial y}(x,H,t)=0$

*(c) $\frac{\partial u}{\partial x}(0,y,t)=0$, $\frac{\partial u}{\partial x}(L,y,t)=0$, $u(x,0,t)=0$, $u(x,H,t)=0$

(d) $u(0,y,t)=0$, $\frac{\partial u}{\partial x}(L,y,t)=0$, $\frac{\partial u}{\partial y}(x,0,t)=0$, $\frac{\partial u}{\partial y}(x,H,t)=0$

(e) $u(0,y,t)=0$, $u(L,y,t)=0$, $u(x,0,t)=0$,
$\frac{\partial u}{\partial y}(x,H,t)+hu(x,H,t)=0$, $(h>0)$

7.3.2. Consider the heat equation in a three-dimensional box-shaped region, $0<x<L$, $0<y<H$, $0<z<W$,

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

subject to the initial condition

$$u(x,y,z,0) = f(x,y,z).$$

Solve the initial value problem and analyze the temperature as $t \to \infty$ if the boundary conditions are

√ (a) $u(0,y,z,t)=0$, $\frac{\partial u}{\partial y}(x,0,z,t)=0$, $\frac{\partial u}{\partial z}(x,y,0,t)=0$,
$u(L,y,z,t)=0$, $\frac{\partial u}{\partial y}(x,H,z,t)=0$, $u(x,y,W,t)=0$

*(b) $\frac{\partial u}{\partial x}(0,y,z,t)=0$, $\frac{\partial u}{\partial y}(x,0,z,t)=0$, $\frac{\partial u}{\partial z}(x,y,0,t)=0$,
$\frac{\partial u}{\partial x}(L,y,z,t)=0$, $\frac{\partial u}{\partial y}(x,H,z,t)=0$, $\frac{\partial u}{\partial z}(x,y,W,t)=0$

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7.3.3 Solve

$$\frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial y^2}$$

on a rectangle ($0 < x < L, 0 < y < H$) subject to

$$u(x,y,0) = f(x,y) \quad \begin{array}{rclrcl} u(0,y,t) & = & 0 & \frac{\partial u}{\partial y}(x,0,t) & = & 0 \\ u(L,y,t) & = & 0 & \frac{\partial u}{\partial y}(x,H,t) & = & 0. \end{array}$$

7.3.4. Consider the wave equation for a vibrating rectangular membrane ($0 < x < L,\ 0 < y < H$)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the initial conditions

$$u(x,y,0) = 0 \text{ and } \frac{\partial u}{\partial t}(x,y,0) = f(x,y).$$

Solve the initial value problem if

(a) $u(0,y,t) = 0, \quad u(L,y,t) = 0, \quad \frac{\partial u}{\partial y}(x,0,t) = 0, \quad \frac{\partial u}{\partial y}(x,H,t) = 0$

*(b) $\frac{\partial u}{\partial x}(0,y,t) = 0, \quad \frac{\partial u}{\partial x}(L,y,t) = 0, \quad \frac{\partial u}{\partial y}(x,0,t) = 0, \quad \frac{\partial u}{\partial y}(x,H,t) = 0$

7.3.5. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - k \frac{\partial u}{\partial t} \quad \text{with } k > 0.$$

(a) Give a *brief* physical interpretation of this equation.

(b) Suppose that $u(x,y,t) = f(x)g(y)h(t)$. What ordinary differential equations are satisfied by f, g, and h?

7.3.6. Consider Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in a right cylinder whose base is arbitrarily shaped (see Fig. 7.3.3). The top is $z = H$ and the bottom is $z = 0$. Assume that

$$\begin{array}{rcl} \frac{\partial}{\partial z} u(x,y,0) & = & 0 \\ u(x,y,H) & = & f(x,y) \end{array}$$

and $u = 0$ on the "lateral" sides.

(a) Separate the z-variable in general.

*(b) Solve for $u(x,y,z)$ if the region is a rectangular box, $0 < x < L, 0 < y < W, 0 < z < H$.

this case, (7.4.11), the generalized Fourier coefficient a_{nm} can be evaluated in two equivalent ways:

(a) Using one two-dimensional orthogonality formula for the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$

(b) Using two one-dimensional orthogonality formulas

4b. **Convergence.** As with any Sturm-Liouville eigenvalue problem (see Sec. 5.10), a *finite* series of the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$ may be used to approximate a function $f(x, y)$. In particular, we could show that if we measure error in the mean-square sense,

$$E \equiv \iint_R \left(f - \sum_\lambda a_\lambda \phi_\lambda \right)^2 dx\, dy, \tag{7.4.16}$$

with weight function 1, then this mean-square error is minimized by the coefficients a_λ being chosen by (7.4.14), the generalized Fourier coefficients. It is known that the approximation improves as the number of terms increases. Furthermore, $E \to 0$ as all the eigenfunctions are included. We say that the series $\sum_\lambda a_\lambda \phi_\lambda$ **converges in the mean to** f.

EXERCISES 7.4

7.4.1. Consider the eigenvalue problem

$$\nabla^2\phi + \lambda\phi = 0$$

$$\begin{aligned} \frac{\partial \phi}{\partial x}(0, y) &= 0 \qquad & \phi(x, 0) &= 0 \\ \frac{\partial \phi}{\partial x}(L, y) &= 0 \qquad & \phi(x, H) &= 0. \end{aligned}$$

*(a) Show that there is a doubly infinite set of eigenvalues.

(b) If $L = H$, show that most eigenvalues have more than one eigenfunction.

(c) Derive that the eigenfunctions are orthogonal in a two-dimensional sense using two one-dimensional orthogonality relations.

7.4.2. Without using the explicit solution of (7.4.7), show that $\lambda \geq 0$ from the Rayleigh quotient, (7.4.6).

7.4.3. If necessary, see Sec. 7.5:

(a) Derive that $\iint (u\nabla^2 v - v\nabla^2 u)\, dx\, dy = \oint (u\nabla v - v\nabla u) \cdot \hat{n}\, ds$.

(b) From part (a), derive (7.4.5).

7.4.4. Derive (7.4.6). If necessary, see Sec. 7.6. [*Hint*: Multiply (7.4.1) by ϕ and integrate.]

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8.2.3. Solve the two-dimensional heat equation with circularly symmetric time-independent sources, boundary conditions, and initial conditions (inside a circle):

$$\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + Q(r)$$

with

$$u(r,0) = f(r) \quad \text{and} \quad u(a,t) = T.$$

8.2.4. Solve the two-dimensional heat equation with time-independent boundary conditions:

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

subject to the boundary conditions

$$\begin{aligned} u(0,y,t) &= 0 \quad \tfrac{\partial}{\partial y}u(x,0,t) = 0 \\ u(L,y,t) &= 0 \quad u(x,H,t) = g(x) \end{aligned}$$

and the initial condition

$$u(x,y,0) = f(x,y).$$

Analyze the limit as $t \to \infty$.

8.2.5. Solve the initial value problem for a two-dimensional heat equation inside a circle (of radius a) with time-independent boundary conditions:

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\nabla^2 u \\ u(a,\theta,t) &= g(\theta) \\ u(r,\theta,0) &= f(r,\theta). \end{aligned}$$

8.2.6. Solve the wave equation with time-independent sources,

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} &= c^2\frac{\partial^2 u}{\partial x^2} + Q(x) \\ u(x,0) &= f(x) \\ \tfrac{\partial}{\partial t}u(x,0) &= g(x), \end{aligned}$$

if an "equilibrium" solution exists. Analyze the behavior for large t. If no equilibrium exists, explain why and reduce the problem to one with homogeneous boundary conditions. Assume that

* (a) $Q(x) = 0$, $u(0,t) = A$, $u(L,t) = B$

(b) $Q(x) = 1$, $u(0,t) = 0$, $u(L,t) = 0$

(c) $Q(x) = 1$, $u(0,t) = A$, $u(L,t) = B$
[*Hint*: Add problems (a) and (b).]

* (d) $Q(x) = \sin\frac{\pi x}{L}$, $u(0,t) = 0$, $u(L,t) = 0$