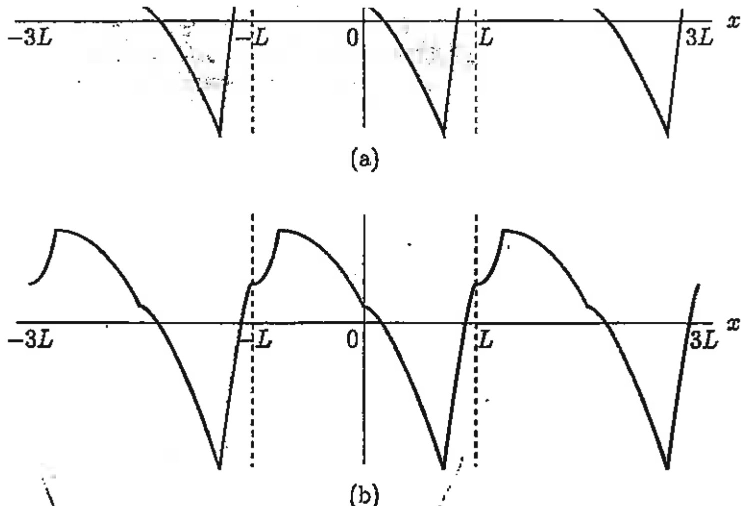

Figure 3.3.17 · Fourier series of a continuous function with (a) $f(-L) \neq f(L)$ and (b) $f(-L) = f(L)$.

ntinuous for $0 \leq x \leq L$ is sketched in Fig. 3.3.18. First we extend $f(x)$ d then periodically. It is easily seen that

> iecewise smooth $f(x)$, the Fourier cosine series of $f(x)$ is continuous converges to $f(x)$ for $0 \leq x \leq L$ if and only if $f(x)$ is continuous.

hat no additional conditions on $f(x)$ are necessary for the cosine series to uous (besides $f(x)$ being continuous). One reason for this result is that if ntinuous for $0 \leq x \leq L$, then the even extension will be continuous for

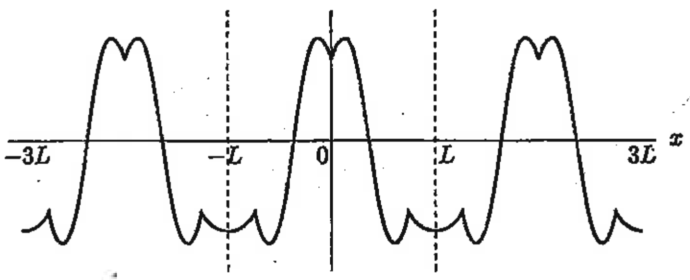

Figure 3.3.18 Fourier cosine series of a continuous function.

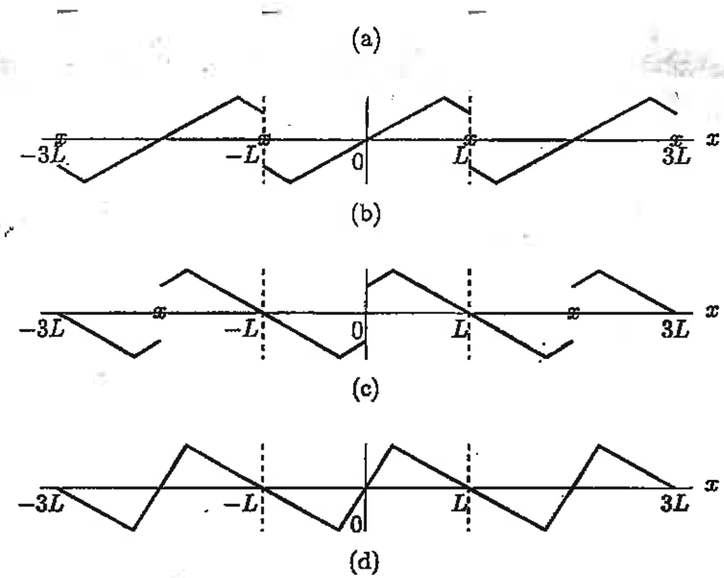

Figure 3.3.19 Fourier sine series of a continuous function with (a) $f(0) \neq 0$ and $f(L) \neq 0$; (b) $f(0) = 0$ but $f(L) \neq 0$; (c) $f(L) = 0$ but $f(0) \neq 0$; and (d) $f(0) = 0$ and $f(L) = 0$.

$-L \leq x \leq L$. Also note that the even extension is the same at $\pm L$. Thus, the periodic extension will automatically be continuous at the endpoints.

Compare this result to what happens for a Fourier sine series. Four examples are considered in Fig. 3.3.19, all continuous functions for $0 \leq x \leq L$. From the first three figures, we see that it is possible for the Fourier sine series of a continuous function to be discontinuous. It is seen that

> For piecewise smooth functions $f(x)$, the Fourier sine series of $f(x)$ is continuous and converges to $f(x)$ for $0 \leq x \leq L$ if and only if $f(x)$ is continuous and both $f(0) = 0$ *and* $f(L) = 0$.

If $f(0) \neq 0$, then the odd extension of $f(x)$ will have a jump discontinuity at $x = 0$, as illustrated in Figs. 3.3.19a and c. If $f(L) \neq 0$, then the odd extension at $x = -L$ will be of opposite sign from $f(L)$. Thus, the periodic extension will not be continuous at the endpoints if $f(L) \neq 0$ as in Figs. 3.3.19a and b.

EXERCISES 3.3

3.3.1. For the following functions, sketch $f(x)$, the Fourier series of $f(x)$, the Fourier sine series of $f(x)$, and the Fourier cosine series of $f(x)$.

(a) $f(x) = 1$

(b) $f(x) = 1 + x$

(c) $f(x) = \begin{cases} x & x < 0 \\ 1 + x & x > 0 \end{cases}$

*(d) $f(x) = e^x$

(e) $f(x) = \begin{cases} 2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$

3.3.2. For the following functions, sketch the Fourier sine series of $f(x)$ and determine its Fourier coefficients.

(a) $f(x) = \cos \pi x / L$ [Verify formula (3.3.13).]

(b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

*(d) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

3.3.3. For the following functions, sketch the Fourier sine series of $f(x)$. Also, roughly sketch the sum of a *finite* number of nonzero terms (at least the first two) of the Fourier sine series:

(a) $f(x) = \cos \pi x / L$ [Use formula (3.3.13).]

(b) $f(x) = \begin{cases} 1 & x < L/2 \\ 0 & x > L/2 \end{cases}$

(c) $f(x) = x$ [Use formula (3.3.12).]

3.3.4. Sketch the Fourier cosine series of $f(x) = \sin \pi x / L$. Briefly discuss.

3.3.5. For the following functions, sketch the Fourier cosine series of $f(x)$ and determine its Fourier coefficients:

(a) $f(x) = x^2$ (b) $f(x) = \begin{cases} 1 & x < L/6 \\ 3 & L/6 < x < L/2 \\ 0 & x > L/2 \end{cases}$ (c) $f(x) = \begin{cases} 0 & x < L/2 \\ x & x > L/2 \end{cases}$

3.3.6. For the following functions, sketch the Fourier cosine series of $f(x)$. Also, roughly sketch the sum of a finite number of nonzero terms (at least the first two) of the Fourier cosine series:

(a) $f(x) = x$ [Use formulas (3.3.22) and (3.3.23).]

(b) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ [Use carefully formulas (3.2.6) and (3.2.7).]

(c) $f(x) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ [*Hint*: Add the functions in parts (b) and (c).]

3.3.8. (a) Determine formulas for the even extension of any f the formula for the even part of $f(x)$.

(b) Do the same for the odd extension of $f(x)$ and the c

(c) Calculate and sketch the four functions of parts (a)

$$f(x) = \begin{cases} x & x > 0 \\ x^2 & x < 0. \end{cases}$$

Graphically add the even and odd parts of $f(x)$. Wl larly, add the even and odd extensions. What occurs

3.3.9. What is the sum of the Fourier sine series of $f(x)$ and t series of $f(x)$? [What is the sum of the even and odd ext

*3.3.10. If $f(x) = \begin{cases} x^2 & x < 0 \\ e^{-x} & x > 0 \end{cases}$, what are the even and odd pa

3.3.11. Given a sketch of $f(x)$, describe a procedure to sketch t parts of $f(x)$.

3.3.12. (a) Graphically show that the even terms (n even) of the of any function on $0 \leq x \leq L$ are odd (antisymmetric)

(b) Consider a function $f(x)$ that is odd around $x = L/$ odd coefficients (n odd) of the Fourier sine series of f are zero.

*3.3.13. Consider a function $f(x)$ that is even around $x = L/2$. Sh coefficients (n even) of the Fourier sine series of $f(x)$ on 0

3.3.14. (a) Consider a function $f(x)$ that is even around $x =$ the odd coefficients (n odd) of the Fourier cosine $0 \leq x \leq L$ are zero.

(b) Explain the result of part (a) by considering a Fouri $f(x)$ on the interval $0 \leq x \leq L/2$.

3.3.15. Consider a function $f(x)$ that is odd around $x = L/2$. Sh coefficients (n even) of the Fourier cosine series of $f(x)$ c zero.

3.3.16. Fourier series can be defined on other intervals besides – pose that $g(y)$ is defined for $a \leq y \leq b$. Represent g(trigonometric functions with period $b - a$. Determine form ficients. [*Hint*: Use the linear transformation

$$y = \frac{a+b}{2} + \frac{b-a}{2L}x.]$$

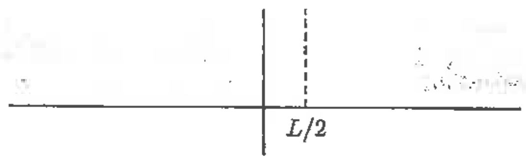

Figure 3.2.1 Sketch of $f(x)$

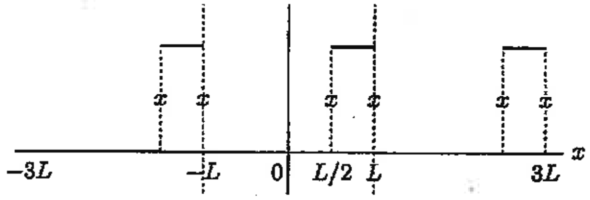

Figure 3.2.2 Fourier series of $f(x)$.

ɔle,

$$\text{ɩ}_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} = \begin{cases} \frac{1}{2} & x = -L \\ 0 & -L < x < L/2 \\ \frac{1}{2} & x = L/2 \\ 1 & L/2 < x < L \\ \frac{1}{2} & x = L. \end{cases}$$

ies can converge to rather strange functions, but they are not so different riginal function.

er coefficients. For a given $f(x)$, it is *not* necessary to calculate : coefficients in order to sketch the Fourier series of $f(x)$. However, it is to know how to calculate the Fourier coefficients, given by (3.2.2). The of Fourier coefficients can be an algebraically involved process. Some- an exercise in the method of integration by parts. Often, calculations plified by judiciously using integral tables or computer algebra systems. ıt, we can always use a computer to approximate the coefficients numer- ın overly simple example but one that illustrates some important points, x) given by (3.2.5). From (3.2.2), the coefficients are

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{L/2}^{L} dx = \frac{1}{4} \tag{3.2.6}$$

$$= \frac{1}{n\pi}\left(\sin n\pi - \sin\frac{n\pi}{2}\right) \tag{3.2.7}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\int_{L/2}^{L} \sin\frac{n\pi x}{L}\,dx = \frac{-1}{n\pi}\cos\frac{n\pi x}{L}\Big|_{L/2}^{L}$$

$$= \frac{1}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right). \tag{3.2.8}$$

We omit simplifications that arise by noting that $\sin n\pi = 0$, $\cos n\pi = (-1)^n$, and so on.

EXERCISES 3.2

3.2.1. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$). Compare $f(x)$ to its Fourier series:

(a) $f(x) = 1$

*(b) $f(x) = x^2$

(c) $f(x) = 1 + x$

*(d) $f(x) = e^x$

(e) $f(x) = \begin{cases} x & x < 0 \\ 2x & x > 0 \end{cases}$

*(f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 + x & x > 0 \end{cases}$

(g) $f(x) = \begin{cases} x & x < L/2 \\ 0 & x > L/2 \end{cases}$

3.2.2. For the following functions, sketch the Fourier series of $f(x)$ (on the interval $-L \le x \le L$) and determine the Fourier coefficients:

*(a) $f(x) = x$

(b) $f(x) = e^{-x}$

*(c) $f(x) = \sin\frac{\pi x}{L}$

(d) $f(x) = \begin{cases} 0 & x < 0 \\ x & x > 0 \end{cases}$

(e) $f(x) = \begin{cases} 1 & |x| < L/2 \\ 0 & |x| > L/2 \end{cases}$

*(f) $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$

(g) $f(x) = \begin{cases} 1 & x < 0 \\ 2 & x > 0 \end{cases}$

3.4.3. Suppose that $f(x)$ is continuous [except for a jump discontinuity at $x = x_0$, $f(x_0^-) = \alpha$ and $f(x_0^+) = \beta$] and df/dx is piecewise smooth.

*(a) Determine the Fourier sine series of df/dx in terms of the Fourier cosine series coefficients of $f(x)$.

(b) Determine the Fourier cosine series of df/dx in terms of the Fourier sine series coefficients of $f(x)$.

3.4.4. Suppose that $f(x)$ and df/dx are piecewise smooth.

(a) Prove that the Fourier sine series of a continuous function $f(x)$ can only be differentiated term by term if $f(0) = 0$ and $f(L) = 0$.

(b) Prove that the Fourier cosine series of a continuous function $f(x)$ can be differentiated term by term.

3.4.5. Using (3.3.13) determine the Fourier cosine series of $\sin \pi x/L$.

3.4.6. There are some things wrong in the following demonstration. Find the mistakes and correct them.

In this exercise we attempt to obtain the Fourier cosine coefficients of e^x:

$$e^x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}. \tag{3.4.22}$$

Differentiating yields

$$e^x = -\sum_{n=1}^{\infty} \frac{n\pi}{L} A_n \sin \frac{n\pi x}{L},$$

the Fourier sine series of e^x. Differentiating again yields

$$e^x = -\sum_{n=1}^{\infty} \left(\frac{n\pi}{L}\right)^2 A_n \cos \frac{n\pi x}{L}. \tag{3.4.23}$$

Since equations (3.4.22) and (3.4.23) give the Fourier cosine series of e^x, they must be identical. Thus,

$$\left.\begin{array}{l} A_0 = 0 \\ A_n = 0 \end{array}\right\} \text{ (obviously wrong!).}$$

By correcting the mistakes, you should be able to obtain A_0 and A_n *without* using the typical technique, that is, $A_n = 2/L \int_0^L e^x \cos n\pi x/L \, dx$.

3.4.7. Prove that the Fourier series of a continuous function $u(x,t)$ can be differentiated term by term with respect to the parameter t if $\partial u/\partial t$ is piecewise smooth.

EXERCISES 3.5

3.5.1. Consider

$$x^2 \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}. \tag{3.5.12}$$

(a) Determine b_n from (3.3.11), (3.3.12), and (3.5.6).

(b) For what values of x is (3.5.12) an equality?

*(c) Derive the Fourier cosine series for x^3 from (3.5.12). For what values of x will this be an equality?

3.5.2. (a) Using (3.3.11) and (3.3.12), obtain the Fourier cosine series of x^2.

(b) From part (a), determine the Fourier sine series of x^3.

3.5.3. Generalize Exercise 3.5.2, in order to derive the Fourier sine series of x^m, m odd.

*3.5.4. Suppose that $\cosh x \sim \sum_{n=1}^{\infty} b_n \sin n\pi x/L$.

(a) Determine b_n by correctly differentiating this series twice.

(b) Determine b_n by integrating this series twice.

3.5.5. Show that B_n in (3.5.9) satisfies $B_n = a_n/(n\pi/L)$, where a_n is defined by (3.5.1).

3.5.6. Evaluate

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots$$

by evaluating (3.5.5) at $x = 0$.

*3.5.7. Evaluate

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots$$

using (3.5.6).

3.6 Complex Form of Fourier Series

With periodic boundary conditions, we have found the theory of Fourier series to be quite useful:

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right), \tag{3.6.1}$$

4.4.3. Consider a slightly damped vibrating string that satisfies

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial t}.$$

(a) Briefly explain why $\beta > 0$.

*(b) Determine the solution (by separation of variables) that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

You can assume that this frictional coefficient β is relatively small $(\beta^2 < 4\pi^2 \rho_0 T_0 / L^2)$.

4.4.4. Redo Exercise 4.4.3(b) by the eigenfunction expansion method.

4.4.5. Redo Exercise 4.4.3(b) if $4\pi^2 \rho_0 T_0/L^2 < \beta^2 < 16\pi^2 \rho_0 T_0 / L^2$.

4.4.6. For (4.4.1)–(4.4.3), from (4.4.11) show that

$$u(x,t) = R(x - ct) + S(x + ct),$$

where R and S are some functions.

4.4.7. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially at rest, $g(x) = 0$, show that

$$u(x,t) = \frac{1}{2}[F(x - ct) + F(x + ct)],$$

where $F(x)$ is the odd periodic extension of $f(x)$. *Hints:*

1. For all x, $F(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$.
2. $\sin a \cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$.

Comment: This result shows that *the practical difficulty of summing an infinite number of terms of a Fourier series may be avoided for the one-dimensional wave equation.*

4.4.8. If a vibrating string satisfying (4.4.1)–(4.4.3) is initially unperturbed, $f(x) = 0$, with the initial velocity given, show that

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} G(\bar{x})\, d\bar{x},$$

where $G(x)$ is the odd periodic extension of $g(x)$. *Hints:*

1. For all x, $G(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$.