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Another Return of "Return to Equilibrium"

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Abstract: The property of "*return to equilibrium*" is established for a class of quantum-mechanical models describing interactions of a (toy) atom with black-body radiation, or of a spin with a heat bath of scalar bosons, under the assumption that the interaction strength is *sufficiently weak*. For models describing the first class of systems, our upper bound on the interaction strength is *independent* of the temperature T, (with $0 < T \leq T_0 < \infty$), while, for the spin-boson model, it tends to zero logarithmically, as $T \to 0$. Our result holds for interaction form factors with physically realistic infrared behaviour.

Three key ingredients of our analysis are: a suitable concrete form of the Araki-Woods representation of the radiation field, Mourre's positive commutator method combined with a recent virial theorem, and a norm bound on the difference between the equilibrium states of the interacting and the non-interacting system (which, for the system of an atom coupled to black-body radiation, is valid for *all* temperatures $T \geq 0$, assuming only that the interaction strength is sufficiently weak).

1. Introduction

The problem of *return to equilibrium* for models describing small systems with finitely many degrees of freedom coupled to a dispersive heat bath at positive temperature has been studied at various levels of mathematical precision, since the early days of quantum theory. Fairly recently, a new approach to this problem based on spectral theory for thermal Hamiltonians, or Liouvillians, has been described and applied to simple models in [18, 19]. The general *strategy* followed in our paper is based on the spectral approach proposed in these references; but our *tactics* are quite different and draw inspiration from techniques developed in [21] that have been motivated by methods in [6]. For further results and methods relevant to our paper, see [5, 8, 15] and, in particular, [12, 13]. The

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work described in *all* these references relies on the deep insights of Haag, Hugenholtz and Winnink [17] and on the Araki-Woods representation [4].

The main result proven in this paper is Theorem 2, asserting *return to equilibrium* for a class of models describing a "small system" with a finite-dimensional state space coupled to a "large system", a dispersive heat bath at some temperature T, with $0 < T \leq T_0 < \infty$. The heat bath is modelled by a spatially infinitely extended free massless bosonic field. The systems we consider fall into two categories corresponding to a *regular* or a *singular* infrared behaviour of the coupling between the two subsystems. Both cases are *physically realistic*.

We show return to equilibrium under the assumption that the interaction strength is sufficiently weak. For infrared-regular systems, such as toy atoms interacting with black-body radiation, our upper bound on the interaction strength only depends on T_0, but not on $T < T_0$. For infrared singular systems, such as the usual spin-boson model, our upper bound on the interaction strength depends on T and tends to zero logarithmically, as $T \to 0$.

The proof of Theorem 2, which is presented in Sect. 3, relies on a result of independent interest, Theorem 3, which says that the norm of the difference of the equilibrium states of the coupled system and of the non-interacting system is small, for weak interaction strength (uniformly in the temperature in the infrared-regular case, and with an explicit temperature-dependent upper bound on the interaction strength for the singular case). Theorem 3 is proven in Sect. 4, and the proof draws on ideas developed in [2, 14, 5, 11].

With Theorems 2 and 3, we accomplish two goals. First, for infrared-regular systems, our results are *uniform* in the temperature T, for $0 < T \leq T_0$ (where the high-temperature bound, T_0, has a clear physical origin, see also the comment after Theorem 2 below), assuming only that the interaction strength is small enough, with an upper bound only depending on T_0. Second, our results also hold for infrared-singular systems, provided the temperature is not too small (depending on the interaction strength).

In order to render our discussion more concrete, we describe the models studied in this paper more explicitly. The first class describes systems consisting of an atom, or of an array of finitely many atoms, coupled to the quantized electromagnetic field. We assume that the temperature T of the electromagnetic field is so small that it is justified to treat the atomic nucleus as *static* and to neglect the role played by atomic states of high energy, in particular those corresponding to the continuous energy spectrum describing an ionized atom. Thus, the upper bound, T_0, on the temperature range considered in this paper is determined by the requirement that

$$k_B T_0 \ll m_{at} c^2, \quad k_B T_0 < \Sigma, \tag{1}$$

where k_B is Boltzmann's constant, $m_{at} c^2$ is the rest energy of an atom, and Σ is the ionization(-threshold) energy. If $T \leq T_0$, with T_0 satisfying (1), then an atom can be described, approximately (in the spirit of the Born-Oppenheimer approximation), as a quantum-mechanical system with a *finite-dimensional state space* spanned by those unperturbed atomic eigenstates corresponding to atomic energies $E \lesssim const\ k_B T_0$ in the discrete spectrum. This defines what we call a "toy (or truncated) atom".

The coupling of the toy atom to the quantized radiation field is described, in the dipole approximation, by an interaction term

$$-e \boldsymbol{d}_{at} \cdot \boldsymbol{E}(\rho), \tag{2}$$

where e is the elementary electric charge, $\boldsymbol{d}_{at}$ is the atomic dipole (moment) operator, and $\boldsymbol{E}$ is the quantized electric field. Furthermore, ρ is a density function corresponding to an approximate (smooth) δ-function peaked at the position of the nucleus and of width comparable to the size of the atom. (The interaction term (2) defines the Ritz Hamiltonian.) When expressed in terms of (Newton-Wigner) photon creation- and annihilation operators the interaction term (2) gives rise to a momentum-space form factor $g_0(k)$ (see Sect. 1.1) corresponding to

$$g_0(k) = i\sqrt{|k|}\widehat{\rho}(k) \propto \sqrt{|k|}, \tag{3}$$

for $|k| \to 0$, where k is the photon momentum. Interactions characterized by an infrared behaviour $g_0(k) \propto |k|^p$, as $|k| \to 0$, with $p > -1/2$, are called *infrared-regular*. Nowhere in our analysis will the *helicity of photons* play an interesting role. The helicity-(polarization-) index will therefore be suppressed in our notation, and we shall think of the heat bath as being described by a scalar field (instead of a transverse vector field).

The second class of models deals with systems of a quantum mechanical spin $\boldsymbol{S}$, with $\boldsymbol{S} \cdot \boldsymbol{S} = s(s+1)$ (and usually $s = 1/2$) coupled to a heat bath described in terms of a quantized, real, massless scalar field φ. Before the spin is coupled to the heat bath it exhibits precession around an external field $\boldsymbol{B}$ pointing in the z-direction. Its dynamics is generated by a Hamiltonian

$$H_{spin} = \epsilon S_z, \quad \text{with} \quad \epsilon \propto |\boldsymbol{B}|. \tag{4}$$

The interactions of the impurity spin with the heat bath give rise to spin-flip processes described by an interaction term e.g. of the form

$$g S_x \varphi(\rho), \tag{5}$$

where g is a coupling constant, and ρ is a density function as described above. The bound, T_0, on the temperature range considered is determined by our desire not to take orbital excitations of the particle (an electron, neutron or atom in a dispersive medium, such as an insulator) carrying the impurity spin $\boldsymbol{S}$ into account.

When φ is expressed in terms of (Newton-Wigner) creation- and annihilation operators the interaction term (5) gives rise to a momentum-space form factor g_0, with

$$g_0(k) = \frac{\widehat{\rho}(k)}{\sqrt{|k|}} \propto \frac{1}{\sqrt{|k|}}, \tag{6}$$

for $|k| \to 0$, where k is the momentum of a scalar boson in the heat bath. Interactions characterized by an infrared behaviour (6) are called *infrared-singular*.

The physical interest of the second model, the *spin-boson model*, is somewhat limited. But it has often been used to illustrate the phenomena of interest to us in this paper.

A general class of model systems reminiscent of the ones just described is introduced, in a formal mathematical way, in Sect. 1.1 below. In the following, we attempt to clarify what we mean by "return to equilibrium". Let $\mathbb{C}^d$ be the state space of the "small system" (the toy atom or impurity spin), and let $\mathcal{B}(\mathbb{C}^d)$ denote the algebra of matrices acting on $\mathbb{C}^d$. Let $\mathfrak{W}$ denote the algebra of Weyl operators over a suitably chosen space of one-boson test functions describing the quantum-mechanical degrees of freedom of the heat bath. The Weyl operators, which are exponentials of field operators smeared out with test functions, are bounded operators, and the algebra $\mathfrak{W}$ they generate is a C^*-algebra. The kinematics of the composed system consisting of the "small system" and the heat bath is described by the C^*-algebra

$$\mathfrak{A} = \mathcal{B}(\mathbb{C}^d) \otimes \mathfrak{W}, \tag{7}$$

and its dynamics, in the Heisenberg picture, is given by a one-parameter group $\{\alpha_t\}$, with $t \in \mathbb{R}$ denoting time, of $*$automorphisms of $\mathfrak{A}$. Before the small system is coupled to the heat bath, $\alpha_t \equiv \alpha_{t,0}$ is given by

$$\alpha_{t,0} = \alpha_t^{at} \otimes \alpha_t^f, \tag{8}$$

where $\alpha_t^{at}(A) = e^{itH_{at}} A e^{-itH_{at}}$, $A \in \mathcal{B}(\mathbb{C}^d)$, is the Heisenberg-picture dynamics of an isolated toy atom, H_{at} is its Hamiltonian, and where α_t^f describes the Heisenberg-picture dynamics of the heat bath. We choose $\{\alpha_t^f\}$ to be the $*$automorphism group of $\mathfrak{W}$ describing the dynamics of free, relativistic, massless bosons, such as photons (but, as announced, we shall suppress reference to their helicity in our notation).

Let ω_β^{at} and ω_β^f be the equilibrium states of the small system isolated from the heat bath, and of the free heat bath, respectively, at inverse temperature $\beta = (k_B T)^{-1}$. Let $\mathcal{H}$ denote the Hilbert space of state vectors of the composed system obtained from the algebra $\mathfrak{A}$ in (7) and the equilibrium state, $\omega_{\beta,0}$, given by

$$\omega_{\beta,0} = \omega_\beta^{at} \otimes \omega_\beta^f, \tag{9}$$

before the small system is coupled to the heat bath, by applying the GNS construction. Furthermore let $\Omega_{\beta,0} \in \mathcal{H}$ denote the cyclic vector in $\mathcal{H}$ corresponding to the state $\omega_{\beta,0}$, and let π_β be the GNS representation of $\mathfrak{A}$ on $\mathcal{H}$. Since $\omega_{\beta,0}$ is time-translation invariant, in the sense that $\omega_{\beta,0}(\alpha_{t,0}(A)) = \omega_{\beta,0}(A)$, for all $A \in \mathfrak{A}$ and all times $t \in \mathbb{R}$, there is a selfadjoint operator, L_0, called thermal Hamiltonian or *Liouvillian*, acting on $\mathcal{H}$ with the properties

$$\pi_\beta(\alpha_{t,0}(A)) = e^{itL_0} \pi_\beta(A) e^{-itL_0}, \tag{10}$$

for all $A \in \mathfrak{A}$, and

$$L_0 \Omega_{\beta,0} = 0. \tag{11}$$

In order to describe interactions between the small system and the heat bath at inverse temperature β, one replaces the (unperturbed) Liouvillian L_0 by an (interacting) Liouvillian L_λ, which is a selfadjoint operator on $\mathcal{H}$ given by

$$L_\lambda = L_0 + \lambda I_\beta, \tag{12}$$

where I_β is an operator on $\mathcal{H}$ determined by a formal interaction Hamiltonian, such as those in (2) or (5). The interaction I_β has the property that the dynamics generated by L_λ defines a $*$automorphism group $\{\sigma_{t,\lambda}\}$ of the von Neumann algebra $\mathfrak{M}_\beta \subset \mathcal{B}(\mathcal{H})$ obtained by taking the weak closure of the algebra $\pi_\beta(\mathfrak{A})$. This means that, for every operator $A \in \mathfrak{M}_\beta$ and arbitrary $t \in \mathbb{R}$, the operator

$$\sigma_{t,\lambda}(A) := e^{itL_\lambda} A e^{-itL_\lambda} \tag{13}$$

belongs again to $\mathfrak{M}_\beta$. (For a representation-independent way of introducing interactions between the small system and the heat bath, see e.g. [12].) Following ideas in [2, 10], one can prove that, for a large class of interactions I_β, there exists a vector $\Omega_{\beta,\lambda} \in \mathcal{H}$ with the property that the state

$$\omega_{\beta,\lambda}(A) := \left\langle \Omega_{\beta,\lambda}, A\Omega_{\beta,\lambda} \right\rangle, \quad A \in \mathfrak{M}_\beta \tag{14}$$

is an *equilibrium state* for the *interacting system*, in the sense that it satisfies the *Kubo-Martin-Schwinger (KMS) condition* for the interacting dynamics on the von Neumann algebra $\mathfrak{M}_\beta$, described by $\sigma_{t,\lambda}$; (see [17], or [19, 5, 10], for an explanation of these notions). The property of return to equilibrium means that the equilibrium state on $\mathfrak{M}_\beta$ given by $\omega_{\beta,\lambda}$ is *dynamically stable*, in the sense of the following definition.

Definition. *The system described by the von Neumann algebra $\mathfrak{M}_\beta$ and the time-evolution $\sigma_{t,\lambda}$ on $\mathfrak{M}_\beta$ (a so-called W^*-dynamical system) has the property of return to equilibrium iff, for an arbitrary normal state ω on $\mathfrak{M}_\beta$ (i.e., a state on $\mathfrak{M}_\beta$ given by a density matrix on $\mathcal{H}$) and an arbitrary operator $A \in \mathfrak{M}_\beta$,*

$$\lim_{t\to\infty} \omega(\sigma_{t,\lambda}(A)) = \omega_{\beta,\lambda}(A), \tag{15}$$

or (more modestly)

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t ds\, \omega(\sigma_{s,\lambda}(A)) = \omega_{\beta,\lambda}(A), \tag{16}$$

(return to equilibrium in the sense of ergodic averages).

The convergence in (15) and (16) follows from the KMS condition for $\omega_{\beta,\lambda}$ and certain spectral properties of the interacting Liouvillian, L_λ; see e.g. [19, 5]. Because $\omega_{\beta,\lambda}$ is invariant under the time evolution $\sigma_{t,\lambda}$, the interaction λI_β in (12) can be chosen s.t.

$$L_\lambda \Omega_{\beta,\lambda} = 0, \tag{17}$$

i.e., zero is an eigenvalue of L_λ. If zero is a *simple* eigenvalue of L_λ then, as a fairly easy consequence of the KMS condition and the von Neumann ergodic theorem, property (16) holds, and if the spectrum of L_λ is absolutely continuous, except for a simple eigenvalue at zero, then (15) holds (this again is easily seen by using the KMS condition and the RAGE theorem. Let us also mention that if the kernel of L_λ is simple then L_λ does not have any nonzero eigenvalues, see e.g. [20]).

The purpose of this paper is to exhibit a class of physically interesting interactions with the property that, for *all* β, with $(k_B T_0)^{-1} \equiv \beta_0 < \beta < \infty$, return to equilibrium in the sense of ergodic averages, (16), holds, provided the coupling constant λ is small enough,

$$0 < |\lambda| < \lambda_0,$$

where, for infrared-regular interactions, λ_0 *only* depends on β_0, while, for infrared-singular interactions, $\lambda_0 \to 0$ logarithmically, as $\beta \to \infty$; see Theorem 2. This result relies, in part, on the following result: Given any $\epsilon > 0$, there exists a positive constant $\lambda_1(\epsilon)$ and a choice of the phases of the vectors $\Omega_{\beta,\lambda}$ and $\Omega_{\beta,0}$ such that

$$\|\Omega_{\beta,\lambda} - \Omega_{\beta,0}\| < \epsilon, \tag{18}$$

for all λ, with $|\lambda| < \lambda_1(\epsilon)$; in the infrared-regular case, the constant $\lambda_1(\epsilon)$ only depends on ϵ, but is *independent* of β, and it decays to zero as $\beta \to \infty$ for infrared-singular systems; see Theorem 3.

A proof of return to equilibrium in the stronger sense (15), and *uniformly* in the temperature $0 < T \leq T_0 < \infty$ has been obtained already in [5] and in [8], under the infrared conditions $g_0(k) \sim |k|^p$, $(|k| \sim 0)$ for some $p > 0$ and $p > 2$, respectively. In addition, [8] show (15) in the infrared-singular case (6), for small coupling, tending to zero as $T \to 0$. The infrared conditions we impose to show (16) are $p = -1/2$ (T-dependent smallness of the coupling), and $p = 1/2, 3/2, p > 2$ (small coupling, uniformly in T).

1.1. The model. We consider a quantum system composed of a "small" subsystem interacting with a "large" subsystem. The pure states of the small subsystem, which is also called *atom* (or *spin*), are given by rays in the finite dimensional Hilbert space

$$\mathcal{H}_{at} = \mathbb{C}^d. \tag{19}$$

The atomic Hamiltonian H_{at} has simple eigenvalues $E_0 < E_1 < \cdots < E_{d-1}$,

$$H_{at} = \mathrm{diag}(E_0, E_1, \ldots, E_{d-1}). \tag{20}$$

It determines the dynamics α_t^{at} of observables $A \in \mathcal{B}(\mathcal{H}_{at})$ according to

$$\alpha_t^{at}(A) = e^{itH_{at}} A e^{-itH_{at}}, \tag{21}$$

where $t \in \mathbb{R}$. For any inverse temperature $0 < \beta < \infty$ there is a unique β-KMS state on $\mathcal{B}(\mathcal{H}_{at})$ associated with the dynamics (21), called the atomic Gibbs state (at inverse temperature β). It is given by

$$\omega_\beta^{at}(\cdot) = \frac{\mathrm{tr}\,\left(e^{-\beta H_{at}}\ \cdot\right)}{\mathrm{tr}\, e^{-\beta H_{at}}}, \tag{22}$$

where the trace is taken over $\mathcal{H}_{at}$.

The large subsystem is infinitely extended and is described by a free, scalar, massless Bose field. Its state is taken to be the equilibrium state at inverse temperature $0 < \beta < \infty$. The description of this state and the GNS representation is standard (see e.g. [4, 18, 19, 12]). We present only the essentials and point out a modification we introduce (namely the phase ϕ in (36)). Let

$$L_0^2 := L^2\big(\mathbb{R}^3, d^3k\big) \cap L^2\big(\mathbb{R}^3, |k|^{-1} d^3k\big) \tag{23}$$

and denote by $\mathfrak{W}(L_0^2)$ the Weyl algebra over L_0^2, i.e., the C^*-algebra generated by Weyl operators $W(f)$, $f \in L_0^2$, satisfying the CCR

$$W(f)W(g) = e^{-\frac{i}{2}\mathrm{Im}\langle f,g\rangle} W(f+g) = e^{-i\mathrm{Im}\langle f,g\rangle} W(g)W(f), \tag{24}$$

and the relations $W(f)^* = W(-f)$, $W(0) = \mathbb{1}$ (unitarity). The brackets $\langle\cdot,\cdot\rangle$ in (24) denote the inner product of $L^2(\mathbb{R}^3, d^3k)$. The large subsystem is described by the β-KMS state ω_β^f on $\mathfrak{W}(L_0^2)$ associated with the dynamics

$$\alpha_t^f(W(f)) = W(e^{it\omega} f), \tag{25}$$

with dispersion relation

$$\omega(k) = |k|. \tag{26}$$

An interaction between the two subsystems can be specified in a representation independent way in terms of a suitable $*$automorphism group $\alpha_{t,\lambda}$ on the C^*-algebra $\mathcal{B}(\mathcal{H}_{at}) \otimes \mathfrak{W}(L^2_0)$, where λ is a perturbation parameter and $\alpha_{t,0} = \alpha_t^{at} \otimes \alpha_t^f$. Here we do not discuss this procedure of defining $\alpha_{t,\lambda}$ – this has been discussed in [12]. Rather, we directly specify how the interacting dynamics acts (is implemented) on the GNS Hilbert space corresponding to

$$\omega_{\beta,0} = \omega_\beta^{at} \otimes \omega_\beta^f, \tag{27}$$

the $(\beta, \alpha_{t,0})$-KMS state on the algebra $\mathfrak{A} = \mathcal{B}(\mathcal{H}_{at}) \otimes \mathfrak{W}(L^2_0)$. The GNS representation of the algebra $\mathfrak{A}$ determined by the state (27) is explicitly given in [4] and has been put, in [18, 19], in a form adapted to the use of the theory of spectral deformations (and of positive commutators). We use a slight modification of the representation in [18, 19]. The representation Hilbert space is

$$\mathcal{H} = \mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F}, \tag{28}$$

where

$$\mathcal{F} = \mathcal{F}\left(L^2(\mathbb{R} \times S^2, du \times d\sigma)\right) \tag{29}$$

is the bosonic Fock space over $L^2(\mathbb{R} \times S^2, du \times d\sigma)$, where $d\sigma$ denotes the uniform measure on S^2. We use the following notational convention: we write $L^2(\mathbb{R} \times S^2)$ for $L^2(\mathbb{R} \times S^2, du \times d\sigma)$ and $L^2(\mathbb{R}^3)$ stands for $L^2(\mathbb{R}^3, d^3k)$, or for $L^2(\mathbb{R}_+ \times S^2, u^2 du \times d\sigma)$ (polar coordinates).

The cyclic vector representing $\omega_{\beta,0}$ in $\mathcal{H}$ is

$$\Omega_{\beta,0} = \Omega_\beta^{at} \otimes \Omega. \tag{30}$$

Here Ω is the vacuum vector in $\mathcal{F}$ and

$$\Omega_\beta^{at} = \left(\operatorname{tr} e^{-\beta H_{at}}\right)^{-1/2} \sum_{j=0}^{d-1} e^{-\beta E_j/2} \varphi_j \otimes \varphi_j, \tag{31}$$

where φ_j is the eigenvector of H_{at} associated to the eigenvalue E_j, see also (20). To complete our description of the GNS representation of (27) we need to give the representation map $\pi_\beta : \mathcal{B}(\mathcal{H}_{at}) \otimes \mathfrak{W}(L^2_0) \to \mathcal{B}(\mathcal{H})$. It is the product

$$\pi_\beta = \pi^{at} \otimes \pi_\beta^f, \tag{32}$$

with

$$\pi^{at}(A) = A \otimes \mathbb{1}_{at}, \tag{33}$$

$$\pi_\beta^f(W(f)) = e^{i\varphi(\tau_\beta f)}, \tag{34}$$

and where, for $h \in L^2(\mathbb{R} \times S^2)$, $\varphi(h)$ is the selfadjoint operator on $\mathcal{F}$ given by

$$\varphi(h) = \frac{a^*(h) + a(h)}{\sqrt{2}}. \tag{35}$$

The operators $a^*(h)$ and $a(h)$ are standard creation and annihilation operators on $\mathcal{F}$, smeared out with h. We take $h \mapsto a^*(h)$ to be linear. The real-linear map $\tau_\beta : L_0^2 \to L^2(\mathbb{R} \times S^2)$ appearing in (34) acts as

$$(\tau_\beta f)(u,\sigma) = \sqrt{\frac{u}{1-e^{-\beta u}}} \begin{cases} \sqrt{u}\, f(u,\sigma), & u > 0, \\ \sqrt{-u}\, e^{i\phi}\overline{f}(-u,\sigma), & u < 0, \end{cases} \tag{36}$$

where we represent f in polar coordinates and $\overline{f}$ means the complex conjugate of f. We have introduced an arbitrary phase $\phi \in \mathbb{R}$ which can be chosen conveniently so as to tune discontinuity properties of the r.h.s. in (36) at $u = 0$. The origin of this freedom can be explained as follows. The expectation functional of ω_β^f is given by

$$L_0^2 \ni f \mapsto \omega_\beta^f(W(f)) = \exp\left[-\frac{1}{4}\int_{\mathbb{R}^3}\left(1+\frac{2}{e^{\beta|k|}-1}\right)|f(k)|^2 d^3k\right], \tag{37}$$

which corresponds to the state of black body radiation at inverse temperature β, see [4]. We define a family of (equivalent) representations of the Weyl algebra $\mathfrak{W}(L_0^2)$ on the Hilbert space (29) by the map

$$\pi_\beta^{U_+,U_-}\big(W(f)\big) = \exp\left[i\varphi\left(\tau_\beta^{U_+,U_-} f\right)\right], \tag{38}$$

where φ is defined in (35), U_+, U_- are arbitrary unitary operators on $L^2(\mathbb{R}^3)$, and

$$\left(\tau_\beta^{U_+,U_-} f\right)(u,\sigma) = \begin{cases} u(U_+(1-e^{-\beta u})^{-1/2} f)(u,\sigma), & u > 0, \\ u(U_-(e^{\beta u}-1)^{-1/2}\overline{f})(-u,\sigma), & u < 0. \end{cases} \tag{39}$$

It is easily seen that, for any choice of the unitaries $U_\pm$,

$$\left\langle \Omega, \exp\left[i\varphi\left(\tau_\beta^{U_+,U_-} f\right)\right]\Omega\right\rangle$$

equals the r.h.s. of (37). Expression (39) reduces to (36) for $U_+ = id$, $U_- = e^{i\phi}$.

Remark. We recall that there is a second representation, $\tilde{\pi}_\beta^{U_+,U_-}$ of $\mathfrak{W}(L_0^2)$ on $\mathcal{F}$ given by

$$\tilde{\pi}_\beta^{U_+,U_-}\big(W(f)\big) = \exp\left[i\varphi\left(\tau_\beta^{U_+,U_-}\left(e^{-\beta u/2} f\right)\right)\right], \tag{40}$$

which commutes with the representation $\pi_\beta^{U_+,U_-}$.

In previous articles involving this setting, [18, 19, 5, 7, 21, 8, 12, 13], the freedom of choosing $U_\pm$ arbitrarily was not used, only $U_\pm = \pm id$ was considered. For a suitable choice of $U_\pm$ one can apply the existing positive commutator methods, based on the generator of translations in $u \in \mathbb{R}$ as conjugate operator, to models with fermionic or bosonic fields having dispersion relation different from (26). These matters will be pursued in another work. Here we restrict our attention to the representation (36), where ϕ is a phase determined by the interaction, see assumption (A1) and the discussion thereafter.

We are now ready to define the interacting dynamics as the $*$automorphism group

$$\sigma_{t,\lambda}(\cdot) = e^{itL_\lambda}(\cdot)e^{-itL_\lambda} \tag{41}$$

on the von Neumann algebra

$$\mathfrak{M}_\beta := \pi_\beta\left(\mathcal{B}(\mathcal{H}_{at}) \otimes \mathfrak{W}(L_0^2)\right)'' \subset \mathcal{B}(\mathcal{H}), \tag{42}$$

where $''$ denotes the double commutant (weak closure), and where the generator L_λ, called the *standard Liouvillian* of the system, is the selfadjoint operator on $\mathcal{H}$ given by ([18, 19, 12])

$$L_\lambda = L_0 + \lambda I, \tag{43}$$

with

$$L_0 = L_{at} + L_f, \quad L_{at} = H_{at} \otimes \mathbb{1}_{at} - \mathbb{1}_{at} \otimes H_{at}, \quad L_f = \mathrm{d}\Gamma(u). \tag{44}$$

Here, $\mathrm{d}\Gamma(u)$ denotes the second quantization (acting on $\mathcal{F}$) of the operator of multiplication by $u \in \mathbb{R}$, λ is a coupling constant, and I is the finite sum

$$I = \sum_\alpha \left\{G_\alpha \otimes \mathbb{1}_{at} \otimes \varphi(\tau_\beta(g_\alpha)) - \mathbb{1}_{at} \otimes \mathcal{C}_{at} G_\alpha \mathcal{C}_{at} \otimes \varphi(\tau_\beta(e^{-\beta u/2} g_\alpha))\right\}, \tag{45}$$

where the operators G_α are bounded, selfadjoint operators on $\mathcal{H}_{at}$, and the functions $g_\alpha \in L_0^2$ are called *form factors*. $\mathcal{C}_{at}$ is the antilinear operator of component-wise complex conjugation in the basis $\{\varphi_j\}_{j=0}^{d-1}$ diagonalizing H_{at}. Note that L_0 does not depend on the choice of the phase ϕ, but I does. The following relative bounds are standard:

$$\|I(N+1)^{-1/2}\|, \ \|(N+1)^{-1/2} I\| < C(1+1/\beta), \tag{46}$$

where C is some constant which is independent of β.

At temperature zero ($\beta = \infty$), the Liouvillian (43) corresponds to the Hamiltonian

$$H_\lambda = H_{at} + \mathrm{d}\Gamma(\omega) + \lambda \sum_\alpha G_\alpha \otimes \varphi(g_\alpha), \tag{47}$$

which describes interactions of the atom with the quantized field involving emission and absorption of field quanta.

The pair $(\mathfrak{M}_\beta, \sigma_{t,\lambda})$ is called a W^*-dynamical system. For $\lambda = 0$ the state on $\mathfrak{M}_\beta$ determined by $\Omega_{\beta,0}$ is a $(\beta, \sigma_{t,0})$-KMS state. It is well known ([2, 14, 5, 10]) that the vector

$$\Omega_{\beta,\lambda} := Z_{\beta,\lambda}^{-1}\, e^{-\beta(L_0 + \lambda I_\ell)/2} \Omega_{\beta,0} \in \mathcal{H}, \tag{48}$$

where $Z_{\beta,\lambda}$ is a normalization factor, and I_ℓ is obtained from I by dropping the second term in the sum (45), defines a $(\beta, \sigma_{t,\lambda})$-KMS state on $\mathfrak{M}_\beta$.

Before stating our results we make two assumptions on the interaction.

(A1) The form factors are given by $g_\alpha(u,\sigma) = u^p \widetilde{g}_\alpha(u,\sigma)$, where p takes one of the values $-1/2$, $1/2$, $3/2$ or $p > 2$, and the $\widetilde{g}_\alpha$ satisfy a set of conditions we describe next. For fixed σ and α, the map $u \mapsto \widetilde{g}_\alpha(u,\sigma)$ is C^3 on $(0,\infty)$ and

$$\|\partial_u^j \widetilde{g}_\alpha\|_{L^2(\mathbb{R}^3)} < \infty, \ \text{for } j = 0, 1, 2, 3. \tag{49}$$

If $p = -1/2$, $1/2$ or $3/2$ then the limits

$$\partial_u^j \widetilde{g}_\alpha(0,\sigma) := \lim_{u \to 0_+} \partial_u^j \widetilde{g}_\alpha(u,\sigma) \tag{50}$$

exist, for $j = 0, 1, 2$, and there is a phase $\phi_0 \in \mathbb{R}$, not depending on α, σ and $j = 0, 1, 2$, s.t.

$$e^{-i\phi_0} \partial_u^j \widetilde{g}_\alpha(0,\sigma) \in \mathbb{R}. \tag{51}$$

In addition, if $p = -1/2, 1/2$ then we require $\partial_u \widetilde{g}_\alpha(0,\sigma) = 0$. Finally, we assume that

$$\|u^2 g_\alpha\|_{L^2(\mathbb{R}^3)} < \infty. \tag{52}$$

(A2) It is assumed that

$$\min_{E_m \neq E_n} \int_{S^2} d\sigma \left| \sum_\alpha \langle \varphi_m, G_\alpha \varphi_n \rangle \, g_\alpha\left(|E_m - E_n|, \sigma\right) \right|^2 > 0. \tag{53}$$

Discussion of Assumptions (A1) and (A2). Assumption (A1) concerns smoothness and decay properties of the form factors, which are necessary in the application of the Virial Theorem, see the remark after Theorem 5. If the interaction is characterized, according to (A1), by $p = -1/2$, then we choose the phase ϕ in (36) to be $\phi = 2\phi_0$. For all other values of p we take $\phi = \pi + 2\phi_0$. For $p = -1/2, 1/2$ an admissible infrared behaviour of the form factors is $g_\alpha \sim u^p$ times a constant, as $u \sim 0$. Other than for the applicability of the Virial Theorem, condition (52) is also used to show that L_λ is selfadjoint (for any $\lambda \in \mathbb{R}$). This follows from the Glimm-Jaffe-Nelson commutator theorem, see [12].

Assumption (A2) is called the *Fermi Golden Rule Condition* and has been discussed extensively in previous works, see e.g. [18, 19, 5, 7, 21, 8, 9]. Its role is to guarantee that the probability of absorption and emission processes of field quanta does not vanish in second order perturbation theory (in λ). This can be translated into a suitable positivity condition on an operator Γ_0, called the *level shift operator*, see (56) below. Let

$$\Pi = P_0 \otimes P_\Omega \tag{54}$$

denote the projection onto the kernel of L_0, where P_0 is the rank-d projection onto the kernel of L_{at}, and P_Ω is the projection onto $\mathbb{C}\Omega$, Ω being the vacuum vector in $\mathcal{F}$, see (29). We will see that if the non-negative operator $\Pi I \delta(L_0) I \Pi$, where δ is the Dirac distribution, has a one-dimensional kernel (the dimension is at least one, since the kernel contains the atomic Gibbs state (31)) then the system has the property of return to equilibrium.

Theorem 1. *Assume (53). There is an $\epsilon_0 > 0$, independent of $\beta \geq \beta_0$ (for any β_0 fixed), s.t. if $0 < \epsilon < \epsilon_0$ then*

$$\Pi I \frac{\epsilon}{L_0^2 + \epsilon^2} I \Pi \geq \Gamma_0 \Pi - C\epsilon^{1/4}, \tag{55}$$

where C is a constant independent of β, and Γ_0 is a bounded operator on $\mathcal{H} = \mathcal{H}_{at} \otimes \mathcal{H}_{at} \otimes \mathcal{F}$, acting trivially on the last factor, $\mathcal{F}$, and leaving Ker L_{at} *invariant. Moreover, Γ_0 restricted to* Ker L_{at} *has zero as a simple eigenvalue, with the atomic Gibbs state*

Ω_β^{at} as eigenvector, see (31), and is strictly positive on the complement of $\mathbb{C}\Omega_\beta^{at}$. More precisely, there is a constant $\gamma_0 > 0$, independent of $0 < \beta < \infty$, s.t.

$$\Gamma_0 \upharpoonright_{\mathrm{Ran}\, \overline{P}_{\Omega_\beta^{at}}} \geq \gamma_0. \tag{56}$$

Here, $\overline{P}_{\Omega_\beta^{at}} = \mathbb{1} - P_{\Omega_\beta^{at}}$ and $P_{\Omega_\beta^{at}}$ is the projection onto $\mathbb{C}\Omega_\beta^{at}$.

A proof of this result, in the case where the sum in (45) reduces to a single term, can be in found [21, 5]. It is easy to carry out that proof for the more general interaction (45). An explicit lower bound, in terms of (53), can be given:

$$\Gamma_0 \upharpoonright_{\mathrm{Ran}\, \overline{P}_{\Omega_\beta^{at}}} \geq \min_{E_m \neq E_n} \frac{(E_m - E_m)^2 \operatorname{tr} e^{-\beta H_{at}}}{|e^{-\beta E_m} - e^{-\beta E_n}|} \int_{S^2} \times d\sigma \left| \sum_\alpha \langle \varphi_m, G_\alpha \varphi_n \rangle \, g_\alpha \left(|E_m - E_n|, \sigma \right) \right|^2 ,$$

which yields γ_0 in (56) by minimizing the r.h.s. over $0 < \beta < \infty$.

2. Main Results

Our main result on return to equilibrium is

Theorem 2 (Return to equilibrium). *Assume Conditions (A1) and (A2). There is a constant $\lambda_0 > 0$, independent of $\beta \geq \beta_0$, for any $\beta_0 > 0$ fixed, s.t. if*

$$0 < |\lambda| < \lambda_0 \begin{cases} 1 & \text{if } p > -1/2 \\ \left(1 + \log(1+\beta)\right)^{-9/2} & \text{if } p = -1/2, \end{cases} \tag{57}$$

then the kernel of L_λ is spanned by the interacting KMS vector $\Omega_{\beta,\lambda}$, (48). In other words, the system has the property of return to equilibrium.

If the temperature of the heat bath is very large then second order processes of absorption and emission of field quanta do not dominate the ones of higher order, and we cannot expect to describe the physics of the system using perturbation theory in second order (although, for the *toy atom* considered here, the phenomenon of return to equilibrium is expected to take place at *all* temperatures; see also our discussion in the introduction). This is why, in the following analysis, the dependence of the constant λ_0 in Theorem 2 on β_0 is natural. The fact that, for $p = -1/2$, we must impose an upper bound on the coupling constant tending to zero, as $\beta \to \infty$ (see (57)), can be understood as follows: Our methods are perturbative (in λ) and rely on controlling the (norm-) distance between the KMS states for the interacting and the non-interacting systems (see Theorem 3). One cannot, in general, expect this distance to be small, for small but non-zero coupling constants, uniformly in $\beta \to \infty$. This is due to the fact that, for $p = -1/2$, and in the zero temperature limit, $\beta \to \infty$, the groundstate of an interacting, infrared singular system is *not* in Fock space (i.e., the Hamiltonian (47) does not have a groundstate in $\mathcal{H}_{at} \otimes \mathcal{F}(L^2(\mathbb{R}^3))$, see e.g. [1, 24]), but the non-interacting system ($\lambda = 0$) does have a groundstate in Fock space! Consequently, we expect the difference between the interacting and the non-interacting KMS state to diverge, as $\beta \to \infty$, for $p = -1/2$.

Assuming that the interaction between the small system and the heat bath is such that $\langle \Omega_{\beta,\lambda}, N\Omega_{\beta,\lambda} \rangle$ is small, for small values of λ, uniformly in $\beta \to \infty$, where $N = d\Gamma(\mathbb{1})$ is the number operator in the Araki-Woods representation and $\Omega_{\beta,\lambda}$ is given in (48), then our methods can be used to establish return to equilibrium for sufficiently small values of λ, *uniformly* in $\beta \geq \beta_0$, even when $p = -1/2$.

From a more technical point of view, we can describe the above discussion as follows. A typical estimate involved in our analysis is inequality (46), where C is some finite constant. Then $\|\lambda I(N+1)^{-1/2}\|$ can be made smaller than any constant $\delta > 0$, provided $|\lambda|$ is chosen sufficiently small, *independently* of $\beta > \beta_0$, for an arbitrary, but fixed $\beta_0 > 0$. Similarly, in order to estimate the norm of the difference between the interacting and the non-interacting KMS state, we need an upper bound on the expectation value of the number operator N in the interacting KMS state $\Omega_{\beta,\lambda}$. As explained after the statement of Theorem 3, this expectation value is bounded above by $\|\lambda I_1(N+1)^{-1/2}\|$, where I_1 is defined in (63). For $p = -1/2$, the latter norm is not uniformly bounded in $\beta \geq \beta_0$, but diverges logarithmically, as $\beta \to \infty$. Thus, requiring it to be bounded by a small constant, we must assume that $|\lambda| \log(\beta)$ is sufficiently small.

Among the technical results used in our proof of Theorem 2 we single out the following one, which shows that the perturbed and unperturbed KMS states are close to each other, for small coupling constants. In the infrared-regular regime $p > -1/2$, the difference between the two KMS states is small *independently* of the inverse temperature.

Theorem 3. *Assume (A1) and let $P_{\Omega_{\beta,\lambda}}$ and $P_{\Omega_{\beta,0}}$ denote the projections onto the spans of the interacting and non-interacting KMS states, $\Omega_{\beta,\lambda}$ (see (48)) and $\Omega_{\beta,0}$ (see (30)), respectively. For any $\epsilon > 0$ there is a $\lambda_0(\epsilon) > 0$, which does not depend on $\beta > 0$, s.t. if*

$$|\lambda| < \lambda_0(\epsilon) \begin{cases} 1 & \text{if } p > -1/2 \\ \big(1 + \log(1+\beta)\big)^{-1} & \text{if } p = -1/2 \end{cases} \tag{58}$$

then

$$\left\| P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}} \right\| < \epsilon. \tag{59}$$

Remark. The constant $\lambda_0(\epsilon)$ in Theorem 3 depends on the spectral gap $E_1 - E_0 > 0$ of the atomic Hamiltonian, and, if the norms $\|G_\alpha\|$ are assumed to satisfy a d-independent upper bound, then $\lambda_0(\epsilon)$ can be chosen independently of the dimension d of the atomic Hilbert space.

We prove Theorem 3 in Sect. 4.

3. Proof of Theorem 2

We use a simplified version of the *positive commutator (PC) method*, introduced in the present context, for zero temperature systems, in [6], and extended to the positive temperature situation in [21]. We refer to [7, 8, 22, 15, 16], and to the book [3], for recent different implementations of this method.

3.1. Mechanism of the proof. There are two key ingredients in our proof, the PC estimate and the Virial Theorem. While we give a proof of the PC estimate, we refer to [12] for a proof of the Virial Theorem.

Let $A_f = \mathrm{d}\Gamma(i\partial_u)$ be the second quantization of $i\partial_u$ on $\mathcal{F}$ (cf. (29)) and set

$$A_0 = i\theta\lambda \left(\Pi I \overline{R}_\epsilon^2 - \overline{R}_\epsilon^2 I \Pi \right), \tag{60}$$

where $\Pi = P_0 \otimes P_\Omega$ has been defined in (54), $\overline{R}_\epsilon = \overline{\Pi} R_\epsilon$, $\overline{\Pi} = \mathbb{1} - \Pi$, $R_\epsilon = (L_0^2 + \epsilon^2)^{-1/2}$, and θ, ϵ are positive parameters. We note that A_0 is a bounded operator satisfying $\mathrm{Ran}\, A_0 \subset \mathcal{D}(L_\lambda)$, and that the commutator $[L, A_0]$ extends to a bounded operator with

$$\|[L_\lambda, A_0]\| \le C \left(\frac{\theta|\lambda|}{\epsilon} + \frac{\theta\lambda^2}{\epsilon^2} \right). \tag{61}$$

On the domain $\mathcal{D}(N)$ of the number operator $N = \mathrm{d}\Gamma(\mathbb{1})$ we define the operator

$$B = N + \lambda I_1 + i[L_\lambda, A_0], \tag{62}$$

where

$$I_1 = \sum_\alpha \left(G_\alpha \otimes \mathbb{1}_{at} \otimes \varphi(\partial_u \tau_\beta(g_\alpha)) - \mathbb{1}_{at} \otimes \mathcal{C}_{at} G_\alpha \mathcal{C}_{at} \otimes \varphi(\partial_u e^{-\beta u/2} \tau_\beta(g_\alpha)) \right). \tag{63}$$

The operator B represents the quadratic form $i[L_\lambda, A_f + A_0]$, see [12].

Theorem 4 (Positive commutator estimate). *Assume (A1) and (A2), and fix* $0 < \eta < 2/3$. *For any* $\nu > 1$ *set*

$$\mathfrak{B}_\nu = \{\psi \in \mathcal{D}(N^{1/2}) \mid \|\psi\| = 1, \|(N+1)^{1/2}\psi\| \le \nu\}.$$

There is a choice of the parameters ϵ *and* θ, *and a constant* $\lambda_1(\eta) = \lambda_1 > 0$, *not depending on* ν *and* $\beta \ge \beta_0$, *s.t. if*

$$0 < |\lambda| < \lambda_1 \begin{cases} 1 & \text{if } p > -1/2 \\ \min\left(\frac{1}{1+\log(1+\beta)}, \frac{\nu^{1/\eta - 9/2}}{(1+\log(1+\beta))^\eta} \right) & \text{if } p = -1/2, \end{cases} \tag{64}$$

then we have

$$\overline{P}_{\Omega_{\beta,\lambda}} B \overline{P}_{\Omega_{\beta,\lambda}} \ge |\lambda|^{2-\eta} \nu^{3-9\eta/2} \gamma_0 \overline{P}_{\Omega_{\beta,\lambda}}, \tag{65}$$

in the sense of quadratic forms on $\mathrm{Ran}\, E_\Delta(L_\lambda) \cap \mathfrak{B}_\nu$, *where* Δ *is any interval around the origin s.t.* $\Delta \cap \sigma(L_{at}) = \{0\}$, $E_\Delta(L_\lambda)$ *is the spectral projection, and where* γ_0 *is given in (56).*

We note that it is enough, for our purposes, to examine B as a quadratic form on a subset of $\mathcal{D}(N^{1/2})$, because any eigenvector ψ_λ of L_λ satisfies $\psi_\lambda \in \mathfrak{B}_{\nu_0}$, for some ν_0 which is independent of $|\lambda| \le 1$. Moreover, for $p > -1/2$, ν_0 does not depend on $\beta \ge \beta_0$, while for $p = -1/2$, ν_0 diverges logarithmically for large β. These facts follow from the next result.

Theorem 5 (Regularity of eigenvectors and Virial Theorem, [12, 13]). *Assume (A1). Let ψ_λ be an eigenvector of L_λ. There is a constant $c(p,\beta)<\infty$, not depending on λ, s.t.*

$$\|N^{1/2}\psi_\lambda\| \le c(p,\beta)|\lambda|\,\|\psi_\lambda\|, \tag{66}$$

and s.t. for all $\beta \ge \beta_0$ (for any $\beta_0 > 0$ fixed),

$$c(p,\beta) \le c_1(p)\begin{cases} 1 & \text{if } p > -1/2 \\ 1+\log(1+\beta) & \text{if } p = -1/2 \end{cases}, \tag{67}$$

where c_1 does not depend on $\beta \ge \beta_0$. Moreover,

$$\langle B\rangle_{\psi_\lambda} := \langle \psi_\lambda, B\psi_\lambda\rangle = 0. \tag{68}$$

Remarks. The constant $c(p,\beta)$ can be expressed in terms of the operator I_1 given in (63) as follows:

$$\|I_1(N+1)^{-1/2}\| \le 2\sum_\alpha \|G_\alpha\|\,\|\partial_u\tau_\beta(g_\alpha)\|_{L^2} = c(p,\beta).$$

One can understand (68) formally by expanding the commutator,

$$\big\langle\psi_\lambda, [L_\lambda, A_f + A_0]\psi_\lambda\big\rangle = 2i\,\mathrm{Im}\big\langle L_\lambda\psi_\lambda, (A_f+A_0)\psi_\lambda\big\rangle = 0. \tag{69}$$

The same argument gives $\big\langle [L_\lambda, A_f]\big\rangle_{\psi_\lambda} = 0$, from which it follows that

$$\begin{aligned} 0 \ge \langle N\rangle_{\psi_\lambda} - \big|\langle\lambda I_1\rangle_{\psi_\lambda}\big| &\ge \langle N\rangle_{\psi_\lambda} - c(p,\beta)|\lambda|\,\|\psi_\lambda\|\,\|N^{1/2}\psi_\lambda\| \\ &\ge \frac{1}{2}\langle N\rangle_{\psi_\lambda} - \frac{1}{2}c(p,\beta)^2\lambda^2\|\psi_\lambda\|^2, \end{aligned} \tag{70}$$

which yields the bound (66). In order to make the arguments leading to (68) rigorous, one needs to control multiple commutators of L_λ with $A_f + A_0$ of order up to three. In particular, we need the first, second and third commutator of I with the dilation generator A_f to be a well-defined, relatively $N^{1/2}$-bounded operator, see [21, 12]. The latter condition is satisfied provided

$$\partial_u^j\tau_\beta(g_\alpha) \text{ is continuous in } u\in\mathbb{R} \text{ for } j=0,1,2, \text{ and} \tag{71}$$

$$\partial_u^j\tau_\beta(g_\alpha) \in L^2(\mathbb{R}\times S^2) \text{ for } j=0,1,2,3. \tag{72}$$

We point out that for this argument, i.e. for the proof of (68), the L^2-norms of the functions $\partial_u^j\tau_\beta(g_\alpha)$ do not need to be bounded uniformly in β. It is not difficult to verify that (71), (72) follow from (A1). Let p and ϕ_0 be as in Assumption (A1); then, for $p = 1/2, 3/2$, $p > 2$, we use the representation (36) with $\phi = 2\phi_0$, while for $p = -1/2$, we take $\phi = \pi + 2\phi_0$.

The proof of Theorem 2 is an easy consequence of Theorems 4 and 5. Indeed, if, for λ satisfying (64), with $\nu = \nu_0$ (introduced after Theorem 2), there were a unit eigenvector $\psi_\lambda \in \mathrm{Ker}\, L_\lambda$, orthogonal to $\Omega_{\beta,\lambda}$, then

$$0 = \langle B\rangle_{\psi_\lambda} \ge |\lambda|^{2-\eta}\nu_0^{3-9\eta/2}\gamma_0. \tag{73}$$

Relation (73) cannot hold since the r.h.s. is strictly positive. For $p=-1/2$ condition (64) (with $\nu = \nu_0 = C[1+\log(1+\beta)]$) gives (57), independently of η.

3.2. Proof of Theorem 4. Since $\Omega_{\beta,\lambda}$ is in the kernel of L_λ, the commutator B given in (62) cannot be strictly positive on the entire space; see (68). To show that dim Ker $L_\lambda = 1$ it is natural to try to show that

$$B + \delta P_{\Omega_{\beta,\lambda}} \geq \gamma, \tag{74}$$

for some $\delta \geq \gamma$, where $\gamma > 0$. Let $\Delta \subset \mathbb{R}$ be an interval around the origin not containing any non-zero eigenvalue of the atomic Liouvillian L_{at}. In Subsect. 3.2.1 we prove (74) in the sense of quadratic forms on the spectral subspace of L_0 associated with the interval Δ (see (96)). Using this inequality, we show in Subsect. 3.2.2 that

$$\overline{P}_{\Omega_{\beta,\lambda}} B \overline{P}_{\Omega_{\beta,\lambda}} \geq \frac{1}{2}\gamma \overline{P}_{\Omega_{\beta,\lambda}}, \tag{75}$$

in the sense of quadratic forms on Ran $E_{\Delta'}(L_\lambda) \cap \mathfrak{B}_\nu$, where $E_{\Delta'}(L_\lambda)$ is the spectral projection of L_λ associated to an interval Δ', which can be chosen arbitrarily, as long as it is properly contained in Δ.

3.2.1. PC estimate localized w.r.t. L_0 We will use the Feshbach method with the decomposition

$$\mathcal{H}^0_\Delta := \text{Ran } E^0_\Delta = \text{Ran } E^0_\Delta \Pi \oplus \text{Ran } E^0_\Delta \overline{\Pi}, \tag{76}$$

where Π is given in (54), and where E^0_Δ is the spectral projection of L_0 associated with the interval Δ. For a presentation of this method resembling most closely the form in which it is used here we refer to [21, 12], and, for more background, to [6, 5, 7].

In what follows, C denotes a constant independent of $\lambda, \theta, \epsilon, \beta \geq \beta_0$ (for any fixed $\beta_0 > 0$), and $C(p, \beta)$ denotes a constant independent of $\lambda, \theta, \epsilon$, satisfying the bound given in (67). The values of C, $C(p, \beta)$ can vary from expression to expression.

From $\overline{\Pi} = \overline{P}_0 \otimes P_\Omega + \overline{P}_\Omega$ and the properties of Δ it follows that Ran $E^0_\Delta \overline{\Pi} \subset$ Ran $\overline{P}_\Omega$ and

$$\begin{aligned} E^0_\Delta \overline{\Pi}(B + \delta P_{\Omega_{\beta,\lambda}}) \overline{\Pi} E^0_\Delta &= E^0_\Delta \overline{\Pi} N^{1/2} \left(\mathbb{1} + N^{-1/2} \lambda I_1 N^{-1/2} \right) N^{1/2} \overline{\Pi} E^0_\Delta \\ &\quad + E^0_\Delta \overline{\Pi} \left(i[L_\lambda, A_0] + \delta P_{\Omega_{\beta,\lambda}} \right) \overline{\Pi} E^0_\Delta \\ &\geq \frac{1}{2} E^0_\Delta \overline{\Pi} + E^0_\Delta \overline{\Pi} i[L_\lambda, A_0] \overline{\Pi} E^0_\Delta \\ &\geq \frac{1}{2} \left(1 - C \frac{\theta \lambda^2}{\epsilon^2} \right) E^0_\Delta \overline{\Pi}, \end{aligned} \tag{77}$$

provided

$$\| \overline{P}_\Omega N^{-1/2} \lambda I_1 N^{-1/2} \overline{P}_\Omega \| \leq C(p, \beta) |\lambda| < 1/2, \tag{78}$$

see the remark after Theorem 5, and where we use the bound

$$\| E^0_\Delta \overline{\Pi} [L_\lambda, A_0] \overline{\Pi} E^0_\Delta \| \leq C \frac{\theta \lambda^2}{\epsilon^2} \tag{79}$$

which follows easily from the definition of A_0, (60). We choose the parameters s.t.

$$C \frac{\theta \lambda^2}{\epsilon^2} < 1/2, \tag{80}$$

and hence we have that

$$E^0_\Delta \overline{\Pi}(B+\delta P_{\Omega_{\beta,\lambda}})\overline{\Pi} E^0_\Delta \geq \frac{1}{4} E^0_\Delta \overline{\Pi}. \tag{81}$$

The Feshbach map associated with the decomposition (76) and with the spectral parameter $m < 1/8$, applied to the operator

$$E^0_\Delta (B+\delta P_{\Omega_{\beta,\lambda}}) E^0_\Delta \tag{82}$$

viewed as an operator on the Hilbert space $\mathcal{H}^0_\Delta$, is given by

$$F_{\Pi,m}(E^0_\Delta(B+\delta P_{\Omega_{\beta,\lambda}})E^0_\Delta) = E^0_\Delta \Pi\Big(B+\delta P_{\Omega_{\beta,\lambda}} - (B+\delta P_{\Omega_{\beta,\lambda}}) \\ \times E^0_\Delta \overline{\Pi}\left(\overline{B+\delta P_{\Omega_{\beta,\lambda}}} - m\right)^{-1} \overline{\Pi} E^0_\Delta (B+\delta P_{\Omega_{\beta,\lambda}})\Big)\Pi E^0_\Delta, \tag{83}$$

where the barred operator is understood to be restricted to the subspace Ran $E^0_\Delta \overline{\Pi} \subset \mathcal{H}^0_\Delta$. Using the definition of A_0, (60), and $\Pi I_1 \Pi = 0$, one sees that

$$\Pi B \Pi = 2\theta\lambda^2 \Pi I \overline{R}^2_\epsilon I \Pi \geq 0. \tag{84}$$

We show that the second term on the r.h.s. of (83), which is negative-definite, is smaller than $\Pi B \Pi$. By (81), the norm of the resolvent in (83) is bounded from above by 8 (for $m < 1/8$). Using this fact, the estimates $\|L_0 \overline{R}_\epsilon\| \leq 1$, $\|\overline{R}^2_\epsilon\| \leq \epsilon^{-2}$ and $\overline{\Pi} i[L_\lambda, A_0]\Pi = \theta\lambda\overline{\Pi} L_\lambda \overline{R}^2_\epsilon I \Pi$, we find that, for any $\psi \in \mathcal{H}^0_\Delta$, the modulus of the expectation value $\langle\cdot\rangle_\psi = \langle\psi, \cdot\,\psi\rangle$ of the second term in the r.h.s. of (83) is bounded above by

$$8\|E^0_\Delta \overline{\Pi}(\lambda I_1 + i[L_\lambda, A_0] + \delta P_{\Omega_{\beta,\lambda}})\Pi\psi\|^2 \leq 16\theta^2\lambda^2 \|\overline{R}_\epsilon I \Pi \psi\|^2 \\ + C\left(\delta^2\|\overline{\Pi} P_{\Omega_{\beta,\lambda}}\Pi\|^2 + C(p,\beta)\lambda^2 + \frac{\theta^2\lambda^4}{\epsilon^4}\right)\|\psi\|^2. \tag{85}$$

It follows that

$$\left\langle F_{\Pi,m}(E^0_\Delta(B+\delta P_{\Omega_{\beta,\lambda}})E^0_\Delta)\right\rangle_\psi \geq 2\theta\lambda^2(1-8\theta)\left\langle \Pi I \overline{R}^2_\epsilon I \Pi\right\rangle_\psi + \delta\|P_{\Omega_{\beta,\lambda}}\Pi\psi\|^2 \\ -C\frac{\theta\lambda^2}{\epsilon}\left(\frac{\epsilon}{\theta}C(p,\beta) + \frac{\theta\lambda^2}{\epsilon^3} + \frac{\epsilon}{\theta\lambda^2}\delta^2\|\overline{\Pi}P_{\Omega_{\beta,\lambda}}\Pi\|^2\right)\|\psi\|^2. \tag{86}$$

The expectation value on the r.h.s. of (86) is estimated from below using

$$\Pi I \overline{R}^2_\epsilon I \Pi \geq \frac{1}{\epsilon}\left(\Gamma_0 - C\epsilon^{1/4}\right), \tag{87}$$

provided $\epsilon < \epsilon_0$, see (55), (56). Pick θ and ϵ s.t.

$$\theta < 1/16,\ \epsilon < \epsilon_0, \tag{88}$$

and, for $\psi \in$ Ran Π, note the estimate

$$\theta\lambda^2\left\langle I\overline{R}^2_\epsilon I + \frac{\delta}{\theta\lambda^2}P_{\Omega_{\beta,\lambda}}\right\rangle_\psi \geq \frac{\theta\lambda^2}{\epsilon}\left\langle \gamma_0 \overline{P}_{\Omega^{at}_\beta} + \frac{\epsilon\delta}{\theta\lambda^2}P_{\Omega_{\beta,\lambda}} - C\epsilon^{1/4}\right\rangle_\psi \\ = \frac{\theta\lambda^2}{\epsilon}\gamma_0\left[\left(1 - C\epsilon^{1/4}/\gamma_0\right)\|\psi\|^2 + \left\langle\frac{\epsilon\delta}{\theta\lambda^2\gamma_0}P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}}\right\rangle_\psi\right], \tag{89}$$

where we use that $P_{\Omega_\beta^{at}}\psi = P_{\Omega_{\beta,0}}\psi$ for $\psi \in \mathrm{Ran}\,\Pi$. We choose

$$\delta \geq \frac{\theta\lambda^2}{\epsilon}\gamma_0 \geq \frac{\theta\lambda^2}{4\epsilon}\gamma_0 =: \gamma, \tag{90}$$

see also inequality (74), and

$$C\frac{\epsilon^{1/4}}{\gamma_0} < 1/4. \tag{91}$$

The r.h.s. of (89) is bounded from below by

$$\frac{\theta\lambda^2}{\epsilon}\gamma_0\Big(3/4 - \|P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}}\|\Big)\|\psi\|^2 \geq \frac{\theta\lambda^2}{2\epsilon}\gamma_0\ \|\psi\|^2. \tag{92}$$

In the last step, we have applied Theorem 3, (59), which tells us that $\|P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}}\| < 1/4$, provided

$$\lambda \text{ satisfies the condition (58) (with } \epsilon = 1/4). \tag{93}$$

Combining this with (86), where we use

$$\|\overline{\Pi}P_{\Omega_{\beta,\lambda}}\Pi\|^2 = \|\overline{\Pi}(P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}})\Pi\|^2 \leq \|P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}}\|^2,$$

gives

$$\Big\langle F_{\Pi,m}(E_\Delta^0(B + \delta P_{\Omega_{\beta,\lambda}})E_\Delta^0)\Big\rangle_\psi \geq \frac{\theta\lambda^2}{4\epsilon}\gamma_0\ \|\psi\|^2, \tag{94}$$

provided

$$C\left(\frac{\epsilon}{\theta}C(p,\beta) + \frac{\theta\lambda^2}{\epsilon^3} + \frac{\epsilon\delta^2}{\theta\lambda^2}\right) < \gamma_0/4. \tag{95}$$

The isospectrality property of the Feshbach map tells us that

$$E_\Delta^0(B + \delta P_{\Omega_{\beta,\lambda}})E_\Delta^0 \geq \min\left(\frac{1}{8}, \frac{\theta\lambda^2}{4\epsilon}\gamma_0\right)E_\Delta^0 = \frac{\theta\lambda^2}{4\epsilon}\gamma_0 E_\Delta^0. \tag{96}$$

3.2.2. PC estimate localized w.r.t. L_λ Let $0 \leq \chi_\Delta \leq 1$ be a smooth function with support inside the interval Δ, s.t. $\chi_\Delta(0) = 1$, and denote by $\chi_\Delta^0 = \chi_\Delta(L_0)$ and $\chi_\Delta = \chi_\Delta(L_\lambda)$ the operators obtained from the spectral theorem. We show in this subsection that any unit vector $\psi \in \mathrm{Ran}\,\overline{P}_{\Omega_{\beta,\lambda}} \cap \mathfrak{B}_\nu$, s.t. $\chi_\Delta\psi = \psi$, satisfies

$$\big\langle B + \delta P_{\Omega_{\beta,\lambda}}\big\rangle_\psi = \langle B\rangle_\psi \geq \frac{\theta\lambda^2}{8\epsilon}\gamma_0, \tag{97}$$

provided suitable bounds on the parameters $\epsilon, \lambda, \theta$ are satisfied. We will repeatedly use the estimate

$$\begin{aligned}\|(1-\chi_\Delta^0)\psi\| = \|(\chi_\Delta - \chi_\Delta^0)\psi\| &\leq C|\lambda|\ \|I(N+1)^{-1/2}\|\ \|(N+1)^{1/2}\psi\| \\ &\leq C\nu|\lambda|,\end{aligned} \tag{98}$$

where the first inequality is a consequence of the standard functional calculus. Let us decompose the expectation value

$$\langle B\rangle_\psi = \left\langle \chi^0_\Delta (B+\delta P_{\Omega_{\beta,\lambda}})\chi^0_\Delta\right\rangle_\psi \tag{99}$$

$$+\left\langle (1-\chi^0_\Delta)(B+\delta P_{\Omega_{\beta,\lambda}})(1-\chi^0_\Delta)\right\rangle_\psi \tag{100}$$

$$+2\,\mathrm{Re}\left\langle (1-\chi^0_\Delta)(B+\delta P_{\Omega_{\beta,\lambda}})\chi^0_\Delta\right\rangle_\psi . \tag{101}$$

Because $E^0_\Delta\chi^0_\Delta = \chi^0_\Delta$, inequality (96) implies that

$$\left\langle \chi^0_\Delta (B+\delta P_{\Omega_{\beta,\lambda}})\chi^0_\Delta\right\rangle_\psi \geq \frac{\theta\lambda^2}{4\epsilon}\gamma_0\|\chi^0_\Delta\psi\|^2 \geq \frac{\theta\lambda^2}{4\epsilon}\gamma_0(1-C\nu|\lambda|)\|\psi\|^2. \tag{102}$$

Since $N+\delta P_{\Omega_{\beta,\lambda}}$ is non-negative, we have that

$$\begin{aligned}(100) &\geq -\left|\left\langle (1-\chi^0_\Delta)(\lambda I_1 + i[L_\lambda, A_0])(1-\chi^0_\Delta)\right\rangle_\psi\right| \\ &\geq -|\lambda|\ \|(1-\chi^0_\Delta)\psi\|\ \|I_1(N+1)^{-1/2}\|\ \|(N+1)^{1/2}\psi\| \\ &\quad -\|[L_\lambda, A_0]\|\ \|(1-\chi^0_\Delta)\psi\|^2 \\ &\geq -C\nu^2\frac{\theta\lambda^2}{\epsilon}\left(\frac{\epsilon}{\theta}C(p,\beta)+|\lambda|+\frac{\lambda^2}{\epsilon}\right),\end{aligned} \tag{103}$$

where we have used (61).

Our next task is to estimate (101). Since N commutes (strongly) with χ^0_Δ and $P_{\Omega_{\beta,\lambda}}\psi = 0$, and using that $\psi \in \mathrm{Ran}\,\overline{P}_{\Omega_{\beta,\lambda}}$, we conclude that

$$\begin{aligned}&\mathrm{Re}\left\langle (1-\chi^0_\Delta)(B+\delta P_{\Omega_{\beta,\lambda}})\chi^0_\Delta\right\rangle_\psi \\ &\geq \delta\left\langle (1-\chi^0_\Delta)P_{\Omega_{\beta,\lambda}}(\chi^0_\Delta - 1)\right\rangle_\psi + \mathrm{Re}\left\langle (1-\chi^0_\Delta)(\lambda I_1 + i[L_\lambda, A_0])\chi^0_\Delta\right\rangle_\psi \\ &\geq -\delta\|(1-\chi^0_\Delta)\psi\|^2 - C(p,\beta)\nu\lambda^2\|\psi\|^2 - \left|\left\langle (1-\chi^0_\Delta)[L_\lambda, A_0]\chi^0_\Delta\right\rangle_\psi\right|.\end{aligned} \tag{104}$$

Taking into account that $(1-\chi^0_\Delta)\Pi = 0$ and $\|(1-\chi^0_\Delta)L_0^{-1}\| \leq C$ (the constant is of the size $|\Delta|^{-1}$), one sees that the last term can be estimated as follows:

$$\begin{aligned}&\left|\left\langle (1-\chi^0_\Delta)[L_\lambda, A_0]\chi^0_\Delta\right\rangle_\psi\right| \\ &\quad = \theta|\lambda|\left|\left\langle (1-\chi^0_\Delta)(\lambda I\Pi I\overline{R}^2_\epsilon - L_\lambda\overline{R}^2_\epsilon I\Pi + \lambda\overline{R}^2_\epsilon I\Pi I)\chi^0_\Delta\right\rangle_\psi\right| \\ &\quad \leq C\nu\theta|\lambda|\left(\frac{\lambda^2}{\epsilon^2}+|\lambda|\right)\|\psi\|^2 = C\nu\frac{\theta\lambda^2}{\epsilon}\left(\frac{|\lambda|}{\epsilon}+\epsilon\right)\|\psi\|^2.\end{aligned} \tag{105}$$

Plugging (105) into (104) and combining this with (102), (103), we arrive at the bound

$$\langle B\rangle_\psi \geq \frac{\theta\lambda^2}{4\epsilon}\gamma_0\left((1-C\nu|\lambda|) - \frac{C\nu}{\gamma_0}\left(\nu\frac{\epsilon}{\theta}C(p,\beta)+\nu|\lambda|+\nu\frac{\lambda^2}{\epsilon}+\frac{|\lambda|}{\epsilon}+\epsilon\right)\right)\|\psi\|^2. \tag{106}$$

Inequality (97) then follows by choosing parameters s.t.

$$C\nu|\lambda| < 1/4 \quad \text{and} \quad \frac{C\nu}{\gamma_0}\left(\nu\frac{\epsilon}{\theta}C(p,\beta) + \nu|\lambda| + \nu\frac{\lambda^2}{\epsilon} + \frac{|\lambda|}{\epsilon} + \epsilon\right) < 1/4. \tag{107}$$

3.2.3. Choice of ϵ, θ *and* δ We must show that the conditions

$$(78), (80), (88), (90), (91), (93), (95), (107) \tag{108}$$

can be simultaneously satisfied. We set

$$\lambda = \nu^{-9/2}\lambda', \tag{109}$$

$$\epsilon = \nu^{-3}|\lambda'|^e, \quad \text{some } 0 < e < 1, \tag{110}$$

$$\theta = |\lambda'|^t, \quad \text{some } 0 < t < e < 1 \text{ s.t. } t > 3e - 2, \tag{111}$$

$$\delta = \frac{\theta\lambda^2}{\epsilon}\gamma_0, \tag{112}$$

and it is easily verified that there is a $\lambda_1 > 0$, depending on e, t, but not on ν, $\beta \geq \beta_0$, s.t. if

$$0 < |\lambda| < \lambda_1 \min\left(C(p,\beta)^{-1}, \nu^{1/\eta - 9/2}C(p,\beta)^{-1/\eta}\right), \tag{113}$$

where $\eta = e - t > 0$, then conditions (108) are met. The "gap of the positive commutator" (see (97)) is of size $\frac{\theta\lambda^2}{\epsilon} = |\lambda|^{2-\eta}\nu^{3-9\eta/2}$. The maximal value of η under conditions (110), (111) is taken for $e \to 2/3$, $t \to 0$.

4. Proof of Theorem 3

The following *high-temperature* result is well known. Given any $\epsilon > 0$, there is an $\eta(\epsilon) > 0$ s.t. if

$$\beta|\lambda| < \eta(\epsilon) \tag{114}$$

then inequality (59) in Theorem 3 holds. A proof of this fact can be given by using the explicit expression (48) for the perturbed KMS state, and using the Dyson series expansion to estimate $\|\Omega_{\beta,\lambda} - \Omega_{\beta,0}\|$ (see e.g. [5]). Condition (114) comes from the fact that the term of order λ^n in the Dyson series is given by an integral over an n-fold simplex of size β, and, naively, (114) is needed to ensure that $\|\Omega_{\beta,\lambda} - \Omega_{\beta,0}\|$ is small. We shall improve our estimates on $\|\Omega_{\beta,\lambda} - \Omega_{\beta,0}\|$ by taking advantage of the decay in (imaginary) time of the field propagators.

To start our analysis, we use the fact that the trace-norm majorizes the operator-norm to write

$$\begin{aligned}\|P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}}\|^2 &\leq \|P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}}\|_2^2 = 2\left(1 - \langle\Omega_{\beta,\lambda}, P_{\Omega_{\beta,0}}\Omega_{\beta,\lambda}\rangle\right)\\ &\leq 2\left\langle\Omega_{\beta,\lambda}, \overline{P}_{\Omega_\beta^{at}}\Omega_{\beta,\lambda}\right\rangle + 2\left\langle\Omega_{\beta,\lambda}, \overline{P}_\Omega\Omega_{\beta,\lambda}\right\rangle,\end{aligned} \tag{115}$$

where we use $\mathbb{1} - P_{\Omega_{\beta,0}} \leq \overline{P}_{\Omega_\beta^{at}} + \overline{P}_\Omega$. Here, Ω_β^{at} is the atomic Gibbs state at inverse temperature β given in (31), and Ω is the vacuum vector in $\mathcal{F}$, see (29). We know that

$$\left\langle\Omega_{\beta,\lambda}, \overline{P}_\Omega\Omega_{\beta,\lambda}\right\rangle \leq \|N^{1/2}\Omega_{\beta,\lambda}\|^2 \leq c(p,\beta)^2|\lambda|^2, \tag{116}$$

where $c(p, \beta)$ satisfies (67), see Theorem 5. There is a $\beta_1(\epsilon) \geq \beta_0$ s.t. if $\beta > \beta_1(\epsilon)$ then

$$\| P_{\Omega_\beta^{at}} - P_{\varphi_0 \otimes \varphi_0} \| < \epsilon/2, \tag{117}$$

where φ_0 is the groundstate eigenvector of H_{at} and $P_{\varphi_0 \otimes \varphi_0} \in \mathcal{B}(\mathcal{H}_{at} \otimes \mathcal{H}_{at})$ is the projection onto the span of $\varphi_0 \otimes \varphi_0$. It follows from (115) that

$$\| P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}} \|^2 \leq 2 \left\langle \Omega_{\beta,\lambda}, \overline{P}_{\varphi_0 \otimes \varphi_0} \Omega_{\beta,\lambda} \right\rangle + \epsilon + 2c(p, \beta)^2 |\lambda|^2, \tag{118}$$

for $\beta > \beta_1(\epsilon)$. Let

$$Q = \overline{P}_{\varphi_0} \in \mathcal{B}(\mathcal{H}_{at}) \tag{119}$$

be the projection onto the orthogonal complement of the groundstate subspace of the atomic Hamiltonian H_{at} so that

$$\overline{P}_{\varphi_0 \otimes \varphi_0} \leq Q \otimes \mathbb{1}_{at} + \mathbb{1}_{at} \otimes Q. \tag{120}$$

Noticing that $\left\langle \Omega_{\beta,\lambda}, Q \otimes \mathbb{1}_{at}\ \Omega_{\beta,\lambda} \right\rangle = \left\langle \Omega_{\beta,\lambda}, \mathbb{1}_{at} \otimes Q\ \Omega_{\beta,\lambda} \right\rangle = \omega_{\beta,\lambda}(Q)$ we see from (118) that

$$\| P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}} \|^2 \leq 4\omega_{\beta,\lambda}(Q) + \epsilon + 2c(p, \beta)^2 |\lambda|^2, \tag{121}$$

provided $\beta > \beta_1(\epsilon)$.

Proposition 1. *For any $\epsilon > 0$ there exist $\beta_2(\epsilon) > 0$ and $\lambda_1(\epsilon) > 0$ such that if $\beta > \beta_2(\epsilon)$ and $|\lambda| < \lambda_1(\epsilon)$ then*

$$\omega_{\beta,\lambda}(Q) < \epsilon. \tag{122}$$

The proof is presented below. For now, we use (122) to prove Theorem 3. We set

$$\begin{aligned} \beta_3(\epsilon) &:= \max(\beta_1(\epsilon), \beta_2(\epsilon)), \\ \lambda_0'(\epsilon) &:= \min\left(\lambda_1(\epsilon), c(p, \beta)^{-1} \sqrt{\epsilon/2}, \eta(\epsilon)/\beta_3(\epsilon)\right), \end{aligned} \tag{123}$$

where $\eta(\epsilon)$ is the constant appearing in (114). In the case $p > -1/2$ the constant $c(p, \beta)$ has an upper bound which is uniform in $\beta \geq \beta_0$, see (67), and we take $\lambda_0(\epsilon)$ to be the r.h.s. of (123) with $c(p, \beta)$ replaced by this upper bound. For $p = -1/2$ we can find a $\lambda_0(\epsilon)$, indpendent of $\beta > 0$, satisfying $(1 + \log(1 + \beta))^{-1} \lambda_0(\epsilon) \leq \lambda_0'(\epsilon)$, see (67).

We always assume (58). Inequalities (121) and (122) yield

$$\| P_{\Omega_{\beta,\lambda}} - P_{\Omega_{\beta,0}} \|^2 \leq 6\epsilon, \tag{124}$$

for $\beta > \beta_3(\epsilon)$. If $\beta \leq \beta_3(\epsilon)$ then $\beta|\lambda| < \eta(\epsilon)$, and (59) follows from the high-temperature result mentioned above. This completes the proof of the theorem, given Proposition 1.

Proof of Proposition 1. It is convenient to work with a finite volume approximation

$$\omega_{\beta,\lambda}^{\Lambda}(\cdot) = \frac{\mathrm{tr}\left(e^{-\beta H_\lambda^\Lambda}\,\cdot\right)}{\mathrm{tr}\, e^{-\beta H_\lambda^\Lambda}} \tag{125}$$

of the KMS state $\omega_{\beta,\lambda}$, where $\Lambda = [-L/2, L/2]^3 \subset \mathbb{R}^3$. (We introduce a finite box Λ just in order to be able to make use of some familiar inequalities for traces. The inequalities needed in our proof also hold in the thermodynamic limit, $\Lambda \nearrow \mathbb{R}^3$; but some readers may be less familiar with them.) In (125), the trace is taken over the Hilbert space $\mathcal{H}_{at} \otimes \mathcal{F}(L^2(\Lambda, d^3x))$. For $n = (n_1, n_2, n_3) \in \mathbb{Z}^3$, let

$$e_n^\Lambda(x) = L^{-3/2} e^{2\pi i n x/L}, \;\; E_n^\Lambda = \frac{2\pi}{L}|n| = \frac{2\pi}{L}(n_1^2 + n_2^2 + n_3^2)^{1/2} \tag{126}$$

denote the eigenvectors and eigenvalues of the operator $\sqrt{-\Delta}$ on $L^2(\Lambda, d^3x)$ with periodic boundary conditions at $\partial\Lambda$. We identify the basis $\{e_n^\Lambda\}$ of $L^2(\Lambda^3, d^3x)$ with the canonical basis of $l^2(\mathbb{Z}^3)$, and define the finite-volume Hamiltonian by

$$H_\lambda^\Lambda = H_{at} + H_f^\Lambda + \lambda v^\Lambda, \tag{127}$$

$$v^\Lambda = \sum_\alpha G_\alpha \otimes \varphi(g_\alpha^\Lambda), \tag{128}$$

where $g_\alpha^\Lambda \in l^2(\mathbb{Z}^3)$ is given by

$$g_\alpha^\Lambda(n) = \left(\frac{2\pi}{L}\right)^{3/2} \begin{cases} g_\alpha\left(\frac{2\pi n}{L}\right), & n \neq 0, \\ 1, & n = 0, \end{cases} \tag{129}$$

and the operator

$$H_f^\Lambda = \mathrm{d}\Gamma(h_f^\Lambda), \tag{130}$$

acting on $\mathcal{F}(l^2(\mathbb{Z}^3))$, is the second quantization of the one-particle Hamiltonian

$$h_f^\Lambda e_n^\Lambda = \begin{cases} E_n^\Lambda e_n^\Lambda, & \text{if } n \neq (0,0,0), \\ e_n^\Lambda, & \text{if } n = (0,0,0). \end{cases} \tag{131}$$

On the complement of the zero-mode subspace h_f^Λ equals $\sqrt{-\Delta}$ with periodic boundary conditions. Changing the action of h_f^Λ on finitely many modes (always under the condition that $e^{-\beta H_f^\Lambda}$ is trace-class) does not affect the thermodynamic limit. Similarly, we may modify the definition of g_α^Λ on finitely many modes without altering the thermodynamic limit. The existence of the thermodynamic limit,

$$\lim_{L\to\infty} \omega_{\beta,\lambda}^\Lambda(A) = \omega_{\beta,\lambda}(A), \tag{132}$$

can be proven by expanding $e^{-\beta H_\lambda^\Lambda}$ into a Dyson (perturbation) series and using that

$$\omega_{\beta,0}^\Lambda(A) = \frac{\mathrm{tr}\left(e^{-\beta H_0^\Lambda} A\right)}{\mathrm{tr}\, e^{-\beta H_0^\Lambda}} \tag{133}$$

has the expected thermodynamic limit for quasi-local observables A.

Our goal is to show that $\omega^{\Lambda}_{\beta,\lambda}(Q) < \epsilon$, for Q given in (119), provided β and λ satisfy the conditions given in Proposition 1, *uniformly* in the size of Λ. In what follows, we will use the Hölder and Peierls-Bogoliubov inequalities (see e.g. [23]). The Hölder inequality (for traces) reads

$$\|A_1 \dots A_n\|_1 \le \prod_{j=1}^{n} \|A_j\|_{p_j}, \tag{134}$$

where $1 \le p_j \le \infty$, $\sum_j p_j^{-1} = 1$, and the norms are

$$\|A\|_p = \left(\mathrm{tr}\, |A|^p\right)^{1/p}, \text{ for } p < \infty, \text{ and } \|A\|_\infty = \|A\| \text{ (operator norm)}. \tag{135}$$

The Peierls-Bogoliubov inequality says that

$$\frac{\mathrm{tr}\,\left(e^{A+B}\right)}{\mathrm{tr}\, e^{B}} \ge \exp\left[\mathrm{tr}\,\left(Ae^{B}\right)/\mathrm{tr}\, e^{B}\right], \tag{136}$$

which implies that

$$\frac{\mathrm{tr}\, e^{-\beta H_0^\Lambda}}{\mathrm{tr}\, e^{-\beta H_\lambda^\Lambda}} \le e^{\beta|\lambda\omega^{\Lambda}_{\beta,0}(v^\Lambda)|} = 1, \tag{137}$$

since, by (128), $\omega^{\Lambda}_{\beta,0}(v^\Lambda) = 0$.

Using the Hölder inequality one sees that, for any $0 < \tau \le \beta/2$,

$$\omega^{\Lambda}_{\beta,\lambda}(Q) = \frac{\mathrm{tr}\,\left(e^{-(\beta-2\tau)H_\lambda^\Lambda} e^{-\tau H_\lambda^\Lambda} Q e^{-\tau H_\lambda^\Lambda}\right)}{\mathrm{tr}\, e^{-\beta H_\lambda^\Lambda}} \le \left[\frac{\mathrm{tr}\,\left\{\left(e^{-\tau H_\lambda^\Lambda} Q e^{-\tau H_\lambda^\Lambda}\right)^{\frac{\beta}{2\tau}}\right\}}{\mathrm{tr}\, e^{-\beta H_\lambda^\Lambda}}\right]^{\frac{2\tau}{\beta}}$$

$$= \left[\frac{\mathrm{tr}\,\left\{\left(Q e^{-\frac{\beta}{2M} H_\lambda^\Lambda} Q\right)^{2M}\right\}}{\mathrm{tr}\, e^{-\beta H_\lambda^\Lambda}}\right]^{\frac{1}{2M}}, \tag{138}$$

where we are choosing τ s.t.

$$\frac{\beta}{2\tau} = 2M, \quad \text{for some } M \in \mathbb{N}. \tag{139}$$

Setting

$$v^\Lambda(t) = e^{-tH_0^\Lambda} v^\Lambda e^{tH_0^\Lambda} \tag{140}$$

and using the Dyson series expansion we obtain

$$Q e^{-\frac{\beta}{2M} H_\lambda^\Lambda} Q = A + B, \tag{141}$$

where the selfadjoint operators A and B are given by

$$A = Qe^{-\frac{\beta}{2M}H_0^\Lambda}Q, \tag{142}$$
$$B = \sum_{n\geq 1}(-\lambda)^n \int_{0\leq t_n\leq \ldots \leq t_1 \leq \frac{\beta}{2M}} Qv^\Lambda(t_n)\cdots v^\Lambda(t_1)e^{-\frac{\beta}{2M}H_0^\Lambda}Q\, dt_1\cdots dt_n. \tag{143}$$

We plug (141) into (138), expand $(A+B)^{2M}$ and use the Hölder inequality to arrive at the bound

$$\omega_{\beta,\lambda}^\Lambda(Q) \leq \left[\frac{\mathrm{tr}\,\left(|A|^{2M}\right)}{\mathrm{tr}\,e^{-\beta H_\lambda^\Lambda}}\right]^{\frac{1}{2M}} + \left[\frac{\mathrm{tr}\,\left(|B|^{2M}\right)}{\mathrm{tr}\,e^{-\beta H_\lambda^\Lambda}}\right]^{\frac{1}{2M}}. \tag{144}$$

The first term on the right-hand side of (144) is easy to estimate. Let $\Delta = E_1 - E_0 > 0$ denote the spectral gap of the atomic Hamiltonian H_{at}. Then

$$\frac{\mathrm{tr}\,\left(|A|^{2M}\right)}{\mathrm{tr}\,e^{-\beta H_0^\Lambda}} = \frac{\mathrm{tr}_{\mathcal{H}_{at}}\left(Qe^{-\beta H_{at}}\right)}{\mathrm{tr}_{\mathcal{H}_{at}}e^{-\beta H_{at}}} = \frac{\sum_{j=1}^{d-1}e^{-\beta(E_j-E_0)}}{1+\sum_{j=1}^{d-1}e^{-\beta(E_j-E_0)}}$$
$$\leq \sum_{j=1}^{d-1}e^{-\beta(E_j-E_0)} \leq 2\int_{E_1-E_0}^\infty e^{-\beta x}dx = 2\frac{e^{-\beta\Delta}}{\beta}. \tag{145}$$

Taking into account (137) and (139), we obtain, for $\beta \geq 1$,

$$\left[\frac{\mathrm{tr}\,\left(|A|^{2M}\right)}{\mathrm{tr}\,e^{-\beta H_\lambda^\Lambda}}\right]^{\frac{1}{2M}} \leq 2e^{-2\tau\Delta}. \tag{146}$$

In order to make the r.h.s. small, we take τ large as compared to Δ^{-1} (hence $\beta \geq 2\tau$ must be large enough).

Next, we consider the second term on the r.h.s. of (144). From (137) one sees that

$$\frac{\mathrm{tr}\,\left(|B|^{2M}\right)}{\mathrm{tr}\,e^{-\beta H_\lambda^\Lambda}} \leq \omega_{\beta,0}^\Lambda\left(e^{\beta H_0^\Lambda}|B|^{2M}\right) = \omega_{\beta,0}^\Lambda\left(e^{\beta H_0^\Lambda}B^{2M}\right). \tag{147}$$

We expand

$$e^{\beta H_0^\Lambda}B^{2M} = \sum_{k_1,\ldots,k_{2M}\geq 1} T(k_1,\ldots,k_{2M}), \tag{148}$$

where

$$T(k_1,\ldots,k_{2M}) = (-\lambda)^{k_1+\cdots+k_{2M}}\int_0^{\frac{\beta}{2M}}dt_1^{(1)}\cdots\int_0^{t_{k_1-1}^{(1)}}dt_{k_1}^{(1)}$$
$$\times\int_{\frac{\beta}{2M}}^{2\frac{\beta}{2M}}dt_1^{(2)}\cdots\int_{\frac{\beta}{2M}}^{t_{k_2-1}^{(2)}}dt_{k_2}^{(2)}\ \cdots\ \int_{(2M-1)\frac{\beta}{2M}}^{\beta}dt_1^{(2M)}\cdots\int_{(2M-1)\frac{\beta}{2M}}^{t_{k_{2M}-1}^{(2M)}}dt_{k_{2M}}^{(2M)}$$
$$\times e^{\beta H_0^\Lambda}Qv^\Lambda(t_{k_1}^{(1)})\cdots v^\Lambda(t_1^{(1)})Q\ Qv^\Lambda(t_{k_2}^{(2)})\cdots v^\Lambda(t_1^{(2)})Q\times\cdots$$
$$\cdots\times Qv^\Lambda(t_{k_{2M}}^{(2M)})\cdots v^\Lambda(t_1^{(2M)})Qe^{-\beta H_0^\Lambda}. \tag{149}$$

Note that the time variables in the integrand are ordered,

$$0 \le t_{k_1}^{(1)} \le \cdots \le t_1^{(1)} \le t_{k_2}^{(2)} \le \cdots \le t_1^{(2M)} \le \beta. \tag{150}$$

Our goal is to obtain an upper bound on $|\omega_{\beta,0}^{\Lambda}(T(k_1, \ldots, k_{2M}))|$, sharp enough to show that

$$\sum_{k_1,\ldots,k_{2M}\ge 1} \left|\omega_{\beta,0}^{\Lambda}(T(k_1, \ldots, k_{2M}))\right| \tag{151}$$

converges, and to estimate the value of the series. Note that the factors $e^{\beta H_0^{\Lambda}}$ and $e^{-\beta H_0^{\Lambda}}$ in the integrand in (149) drop when we apply $\omega_{\beta,0}^{\Lambda}$ (cyclicity of the trace), and the expectation value of the integrand in the state $\omega_{\beta,0}^{\Lambda} = \omega_\beta^{at} \otimes \omega_\beta^{f,\Lambda}$ (see (133)) splits into a sum over products

$$\sum_{\alpha_1^{(1)},\ldots,\alpha_{k_1}^{(1)}} \cdots \sum_{\alpha_1^{(2M)},\ldots,\alpha_{k_{2M}}^{(2M)}} \omega_\beta^{at}\left(QG_{\alpha_{k_1}^{(1)}}(t_{k_1}^{(1)})\cdots G_{\alpha_1^{(2M)}}(t_1^{(2M)})Q\right)$$
$$\times \omega_\beta^{f,\Lambda}\left(\varphi_{\alpha_{k_1}^{(1)}}^{\Lambda}(t_{k_1}^{(1)})\cdots \varphi_{\alpha_1^{(2M)}}^{\Lambda}(t_1^{(2M)})\right), \tag{152}$$

where ω_β^{at} and $\omega_\beta^{f,\Lambda}$ are the atomic and field KMS states at inverse temperature β, and

$$G_\alpha(t) = e^{-tH_{at}} G_\alpha e^{tH_{at}}, \tag{153}$$
$$\varphi_\alpha^{\Lambda}(t) = e^{-tH_f^{\Lambda}} \varphi(g_\alpha^{\Lambda}) e^{tH_f^{\Lambda}} = a^*\left(e^{-th_f^{\Lambda}} g_\alpha^{\Lambda}\right) + a\left(e^{th_f^{\Lambda}} g_\alpha^{\Lambda}\right). \tag{154}$$

Using the Hölder inequality (134) it is not difficult to see that

$$\left|\omega_\beta^{at}\left(QG_{\alpha_{k_1}^{(1)}}(t_{k_1}^{(1)})\cdots G_{\alpha_1^{(2M)}}(t_1^{(2M)})Q\right)\right| \le \prod_{j=1}^{2M} \|G_{\alpha_1^{(j)}}\|\cdots\|G_{\alpha_{k_j}^{(j)}}\|. \tag{155}$$

Since $\omega_\beta^{f,\Lambda}$ is a quasi-free state we can estimate the second factor in (152) with the help of Wick's theorem:

$$\omega_\beta^{f,\Lambda}\left(\varphi_{\alpha_1}^{\Lambda}(t_1)\cdots\varphi_{\alpha_{2N}}^{\Lambda}(t_{2N})\right) = \sum_{\mathcal{P}} \prod_{(l,r)\in\mathcal{P}} \omega_\beta^{f,\Lambda}\left(\varphi_{\alpha_l}^{\Lambda}(t_l)\varphi_{\alpha_r}^{\Lambda}(t_r)\right), \tag{156}$$

where the sum extends over all *contraction schemes*, i.e., decompositions of $\{1, \ldots, 2N\}$ into N disjoint, ordered pairs (l, r), $l < r$. Applying (156) to

$$\omega_\beta^{f,\Lambda}\left(\varphi_{\alpha_{k_1}^{(1)}}^{\Lambda}(t_{k_1}^{(1)})\cdots\varphi_{\alpha_1^{(2M)}}^{\Lambda}(t_1^{(2M)})\right) \tag{157}$$

we find that all resulting terms can be organized in *graphs* $\mathcal{G}$, constructed in the following way. Partition the circle of circumference β into $2M$ segments (parametrized by the arc length) $\Delta_j = [(j-1)\frac{\beta}{2M}, j\frac{\beta}{2M}]$, $j = 1, \ldots, 2M$. Put k_j "dots" into the interval Δ_j, each dot representing a time variable $t_\cdot^{(j)} \in \Delta_j$ (increasing times are ordered according to increasing arc length). Pick any dot in any interval and pair it with an arbitrary different dot in any interval. Then pick any unpaired dot (i.e., one not yet paired up) and pair it with any other unpaired dot. Continue this procedure until all dots in all intervals are paired; (notice that the total number of dots on the circle is even, as follows from the gauge-invariance of $\omega_\beta^{f,\Lambda}$). The graph $\mathcal{G}$ associated to such a pairing consists of all pairs – including multiplicity – of intervals (Δ, Δ') with the property that some dot in Δ is paired with some dot in Δ'. "Including multiplicity" means that if, say, three dots of Δ are paired with three dots in Δ', we understand that $\mathcal{G}$ contains the pair (Δ, Δ') three times. The class of all pairings $\mathcal{P}$ leading to a given graph $\mathcal{G}$ is denoted by $C_{\mathcal{G}}$. Let

$$A_{\mathcal{P}} = \prod_{(l,r)\in\mathcal{P}} \omega_\beta^{f,\Lambda}\left(\varphi_{\alpha_l}^\Lambda(t_l)\varphi_{\alpha_r}^\Lambda(t_r)\right) \tag{158}$$

denote the contribution to (156) corresponding to the pairing $\mathcal{P}$. The numerical value, $|\mathcal{G}|$, corresponding to a graph $\mathcal{G}$ is defined by

$$|\mathcal{G}| = \left|\sum_{\mathcal{P}\in C_{\mathcal{G}}} A_{\mathcal{P}}\right|, \tag{159}$$

and it follows from (156), (158) and (159) that

$$\left|\omega_{\beta,f}^\Lambda\left(\varphi_{\alpha_{k_1}^{(1)}}^\Lambda(t_{k_1}^{(1)})\cdots\varphi_{\alpha_1^{(2M)}}^\Lambda(t_1^{(2M)})\right)\right| \le \sum_{\mathcal{G}} |\mathcal{G}|. \tag{160}$$

In order to give an upper bound on the r.h.s. of (160), we must estimate the imaginary-time propagators (two-point functions)

$$\begin{aligned}&\omega_\beta^{f,\Lambda}\left(e^{-t_l H_f^\Lambda}\varphi(g_{\alpha_l}^\Lambda)e^{t_l H_f^\Lambda}e^{-t_r H_f^\Lambda}\varphi(g_{\alpha_r}^\Lambda)e^{t_r H_f^\Lambda}\right)\\ &= \left\langle g_{\alpha_r}^\Lambda, e^{-(\beta+t_l-t_r)h_f^\Lambda}\frac{e^{\beta h_f^\Lambda}}{e^{\beta h_f^\Lambda}-1}g_{\alpha_l}^\Lambda\right\rangle + \left\langle g_{\alpha_l}^\Lambda, e^{-(t_r-t_l)h_f^\Lambda}\frac{e^{\beta h_f^\Lambda}}{e^{\beta h_f^\Lambda}-1}g_{\alpha_r}^\Lambda\right\rangle,\end{aligned} \tag{161}$$

where the $g_{\alpha_{l,r}}^\Lambda \in l^2(\mathbb{Z}^3)$ are given in (129), and where $t_l \in \Delta_l$, $t_r \in \Delta_r$ s.t. $0 \le t_l \le t_r \le \beta$. The r.h.s. of (161) equals

$$\begin{aligned}&\left(\frac{2\pi}{L}\right)^3 \sum_{n\neq(0,0,0)} \Big[\overline{g_{\alpha_r}(2\pi n/L)}g_{\alpha_l}(2\pi n/L)e^{-(\beta+t_l-t_r)E_n^\Lambda}\\ &\qquad + \overline{g_{\alpha_l}(2\pi n/L)}g_{\alpha_r}(2\pi n/L)e^{-(t_r-t_l)E_n^\Lambda}\Big] \times \frac{e^{\beta E_n^\Lambda}}{e^{\beta E_n^\Lambda}-1}\\ &\qquad + \left(\frac{2\pi}{L}\right)^3\left[e^{-(\beta+t_l-t_r)} + e^{-(t_r-t_l)}\right]\frac{e^\beta}{e^\beta - 1}.\end{aligned} \tag{162}$$

In the limit $L \to \infty$, the Riemann sum in (162) converges to

$$\int_{\mathbb{R}^3} d^3k \left[\overline{g_{\alpha_r}(k)}g_{\alpha_l}(k)e^{-(\beta+t_l-t_r)|k|} + \overline{g_{\alpha_l}(k)}g_{\alpha_r}(k)e^{-(t_r-t_l)|k|}\right]\frac{e^{\beta|k|}}{e^{\beta|k|}-1}, \tag{163}$$

since the form factors $g_{\alpha_{l,r}}$ satisfy conditions (A1). The term in (162) coming from $n = (0, 0, 0)$ disappears in the limit $L \to \infty$. (This shows why a redefinition of h_f^Λ on the zero mode does not affect the thermodynamic limit.)

It is not hard to see that, for arbitrary Δ_l, Δ_r and $t_l \in \Delta_l$, $t_r \in \Delta_r$,

$$|t_l - t_r| \geq d_-(\Delta_l, \Delta_r) := \frac{\beta}{2M} \begin{cases} 0, & \text{if } l = r \\ |l - r| - 1, & \text{if } l \neq r \end{cases} \tag{164}$$

and

$$\beta - |t_l - t_r| \geq d_+(\Delta_l, \Delta_r) := \beta - \frac{\beta}{2M} (|l - r| + 1) \,. \tag{165}$$

Defining

$$d(\Delta, \Delta') := \min(d_-(\Delta, \Delta'), d_+(\Delta, \Delta')), \tag{166}$$

we obtain from (161) and (163), and for L large enough,

$$\begin{aligned} &\left|\omega_\beta^{f,\Lambda}\left(e^{-t_l H_f^\Lambda} \varphi(g_{\alpha_l}^\Lambda) e^{t_l H_f^\Lambda} e^{-t_r H_f^\Lambda} \varphi(g_{\alpha_r}^\Lambda) e^{t_r H_f^\Lambda}\right)\right| \\ &\leq 2 \left\langle g_{\alpha_l}^\Lambda, \frac{e^{-d(\Delta_l, \Delta_r)|k|}}{1 - e^{-\beta|k|}} g_{\alpha_l}^\Lambda \right\rangle^{1/2} \left\langle g_{\alpha_r}^\Lambda, \frac{e^{-d(\Delta_l, \Delta_r)|k|}}{1 - e^{-\beta|k|}} g_{\alpha_r}^\Lambda \right\rangle^{1/2} . \end{aligned} \tag{167}$$

Given any two intervals Δ, Δ', set

$$C(\Delta, \Delta') := 4 \max_\alpha \left\langle g_\alpha, \frac{e^{-d(\Delta, \Delta')|k|}}{1 - e^{-\beta|k|}} g_\alpha \right\rangle. \tag{168}$$

If $L \geq C$, for some constant C, then (168) is a volume-independent upper bound on the (finite-volume) two-point functions arising from contractions in the graph expansion (Wick theorem). We are now ready to give an upper bound on the r.h.s. of (160); (see also [11] for similar considerations).

It is useful to start the procedure of pairing dots in the interval with the highest order k. Let π be a permutation of $2M$ objects, s.t.

$$k_{\pi(1)} \geq k_{\pi(2)} \geq \cdots \geq k_{\pi(2M)}. \tag{169}$$

There are $k_{l_1^{(\pi(1))}}$ possibilities of pairing the dot $t_1^{(\pi(1))}$ with some dot in an interval $\Delta_{l_1^{(\pi(1))}}$. We associate to each such pairing the value

$$k_{l_1^{(\pi(1))}} \, C(\Delta_{\pi(1)}, \Delta_{l_1^{(\pi(1))}}) \leq \sqrt{k_{\pi(1)}} \sqrt{k_{l_1^{(\pi(1))}}} \, C(\Delta_{\pi(1)}, \Delta_{l_1^{(\pi(1))}}), \tag{170}$$

where we use (169). Next, we pair the dot labelled by $t_2^{(\pi(1))}$ (if it is still unpaired, otherwise we move to the next unpaired dot) with a dot in $\Delta_{l_2^{(\pi(1))}}$ and associate to this pairing the value

$$\sqrt{k_{\pi(1)}} \sqrt{k_{l_2^{(\pi(1))}}} \, C(\Delta_{\pi(1)}, \Delta_{l_2^{(\pi(1))}}). \tag{171}$$

We continue this procedure until all dots are paired. This yields the estimate

$$\sum_{\mathcal{G}} |\mathcal{G}| \le \prod_{j=1}^{2M} (k_j)^{k_j/2} \sum_{\mathcal{G}} \prod_{(\Delta,\Delta')\in\mathcal{G}} C(\Delta, \Delta'). \tag{172}$$

Next, we establish an upper bound on the sum on the r.h.s. Using that

$$|g_\alpha(k)| \le C|k|^p, \tag{173}$$

for some constant C, and for all α, provided $|k|$ is small enough, with $p > -1$, it is easy to see that

$$C(\Delta, \Delta') \le C \begin{cases} d(\Delta, \Delta')^{-3-2p}, & d(\Delta, \Delta') \neq 0 \\ 1/\beta + 1, & d(\Delta, \Delta') = 0 \end{cases}. \tag{174}$$

Furthermore, using definition (166) and inequality (174), we see that, for any Δ,

$$\sum_{\Delta'} C(\Delta, \Delta') \le \Gamma := C\left(1 + \frac{1}{\beta} + \frac{1}{p+1}\left(\frac{\beta}{2M}\right)^{-2-2p}\right) < \infty, \tag{175}$$

provided $p > -1$. Consequently, we find that

$$\sum_{\mathcal{G}} \prod_{(\Delta,\Delta')\in\mathcal{G}} C(\Delta, \Delta') \le \Gamma^{k_1+\cdots+k_{2M}}. \tag{176}$$

Carrying out the integral over the simplex in (149), and using (152), (155), (160), (172), (176), we obtain the bound

$$\left|\omega^{\Lambda}_{\beta,0}(T(k_1, \ldots, k_{2M}))\right| \le \left(C'|\lambda|\Gamma\frac{\beta}{2M}\right)^{k_1+\cdots+k_{2M}} \prod_{j=1}^{2M} \frac{(k_j)^{k_j/2}}{k_j!}, \tag{177}$$

where $C' = \sum_\alpha \|G_\alpha\|$, and where the factor $(\frac{\beta}{2M})^{k_j}\frac{1}{k_j!}$ is the volume of the simplex $\{t \le t_{k_j} \le \cdots \le t_1 \le t + \frac{\beta}{2M}\}$. Thus, the series (151) converges for all values of λ and $\beta > 0$, and

$$\left[\omega^{\Lambda}_{\beta,0}\left(e^{\beta H_0^\Lambda} B^{2M}\right)\right]^{\frac{1}{2M}} \le C'|\lambda|\Gamma\frac{\beta}{2M} \sum_{k\ge 0}\left(C'|\lambda|\Gamma\frac{\beta}{2M}\right)^k \frac{(k+1)^{\frac{k+1}{2}}}{(k+1)!}. \tag{178}$$

Combining (144), (146), (147), (178), and using (139), we see that if L is large enough (independent of λ or β) then

$$\omega^{\Lambda}_{\beta,\lambda}(Q) \le 2e^{-2\tau\Delta} + C'|\lambda|\Gamma\tau \sum_{k\ge 0}\left(C'|\lambda|\Gamma\tau\right)^k \frac{(k+1)^{\frac{k+1}{2}}}{(k+1)!}. \tag{179}$$

The final step in the proof of Proposition 1 consists in showing that the r.h.s. (which is independent of Λ) can be made arbitrarily small, provided β is large enough and λ is small enough. Pick $\beta_2(\epsilon) > 1$ so large that $e^{-\beta_2(\epsilon)\Delta} < \epsilon/2$. For $\beta \ge \beta_2(\epsilon)$ we choose $\tau = \beta_2(\epsilon)/2 \le \beta/2$. From the definition of Γ, (175), and the relation $\frac{\beta}{2M} = 2\tau$, see (139), we see that $\Gamma\tau \le C(\epsilon)$, uniformly in $\beta \ge \beta_2(\epsilon)$. It follows that there is a $\lambda_1(\epsilon) > 0$ s.t. if $|\lambda| < \lambda_1(\epsilon)$ then the second term on the r.h.s. of (179) is smaller than $\epsilon/2$. This completes the proof of Proposition 1. □

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