Midterm Exam – Solutions

MATH 2260 – Ordinary Differential Equations I

1. Consider the initial value problem

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \qquad y(\pi) = 0, \qquad t > 0.$$

This is a first order linear equation. Its integrating factor is

$$\mu(t) = \exp\left(\int \frac{2}{t}\,dt\right) = t^2, \qquad t > 0.$$

Multiplying the equation by t^2 gives

$$t^2y' + 2ty = \cos t,$$

or

$$(t^2y)' = \cos t.$$

Integrating,

$$t^2y = \sin t + C.$$

Thus

$$y(t) = \frac{\sin t + C}{t^2}.$$

Using $y(\pi) = 0$ gives

$$0 = \frac{\sin \pi + C}{\pi^2},$$

so $C = 0$. Therefore

$$\boxed{y(t) = \frac{\sin t}{t^2}}, \qquad t > 0.$$

2. The differential equation is

$$t(t-4)y' + y = 0, \qquad y(2) = 1.$$

To apply the existence and uniqueness theorem, write the equation in standard form:

$$y' + \frac{1}{t(t-4)}y = 0.$$

The coefficient

$$p(t) = \frac{1}{t(t-4)}$$

is continuous except at

$$t = 0 \qquad \text{and} \qquad t = 4.$$

The initial point $t_0 = 2$ lies between these two singular points. Hence the largest open interval containing 2 on which the coefficient is continuous is

$$\boxed{(0, 4)}.$$

Therefore the solution is certain to exist uniquely on $0 < t < 4$.

3. Let

$$f(y) = e^y - 1.$$

The critical points satisfy

$$e^y - 1 = 0,$$

so the only critical point is

$$\boxed{y = 0}.$$

Since

$$e^y - 1 < 0 \quad \text{for } y < 0, \qquad e^y - 1 > 0 \quad \text{for } y > 0,$$

the sign chart is

y	$(-\infty, 0)$	0	$(0, \infty)$
$f(y)$	$-$	0	$+$

Thus the phase-line arrows point downward for $y < 0$ and upward for $y > 0$. Solutions on both sides move away from the equilibrium. Therefore

$$\boxed{y = 0 \text{ is unstable}}.$$

The graph of $f(y) = e^y - 1$ is strictly increasing, passes through $(0, 0)$, approaches the horizontal asymptote $f = -1$ as $y \to -\infty$, and grows rapidly as $y \to \infty$.

For the solution curves in the ty-plane:

- if $y_0 = 0$, then $y(t) \equiv 0$;
- if $y_0 < 0$, then $y'(t) < 0$, so the solution decreases and moves away from 0;
- if $y_0 > 0$, then $y'(t) > 0$, so the solution increases and moves away from 0.

Hence all nonconstant solutions move away from the equilibrium as t increases.

4. Write

$$M(x, y) = ye^{2xy} + x, \qquad N(x, y) = bxe^{2xy}.$$

The equation is exact if and only if

$$M_y = N_x.$$

We have

$$M_y = e^{2xy} + 2xye^{2xy} = e^{2xy}(1 + 2xy),$$

and

$$N_x = b\left(e^{2xy} + 2xye^{2xy}\right) = be^{2xy}(1 + 2xy).$$

Thus exactness requires

$$\boxed{b = 1}.$$

With $b = 1$, we seek a potential function ψ satisfying

$$\psi_x = ye^{2xy} + x.$$

Integrating with respect to x,

$$\psi(x, y) = \frac{1}{2}e^{2xy} + \frac{1}{2}x^2 + h(y).$$

Then

$$\psi_y = xe^{2xy} + h'(y).$$

Since $\psi_y = N = xe^{2xy}$, it follows that

$$h'(y) = 0.$$

Therefore the implicit solution is

$$\frac{1}{2}e^{2xy} + \frac{1}{2}x^2 = C,$$

or equivalently

$$\boxed{e^{2xy} + x^2 = C}.$$

5. The differential equation is

$$4y'' - y = 0.$$

The characteristic equation is

$$4r^2 - 1 = 0,$$

so

$$r = \frac{1}{2}, \qquad r = -\frac{1}{2}.$$

Hence

$$y(t) = c_1 e^{t/2} + c_2 e^{-t/2}.$$

Using $y(0) = 2$ gives

$$c_1 + c_2 = 2.$$

Also,

$$y'(t) = \frac{1}{2}c_1 e^{t/2} - \frac{1}{2}c_2 e^{-t/2},$$

so $y'(0) = \beta$ gives

$$\frac{1}{2}(c_1 - c_2) = \beta.$$

Therefore

$$c_1 = 1 + \beta, \qquad c_2 = 1 - \beta.$$

Thus the solution of the initial value problem is

$$\boxed{y(t) = (1+\beta)e^{t/2} + (1-\beta)e^{-t/2}}.$$

For $y(t) \to 0$ as $t \to \infty$, the coefficient of the growing exponential $e^{t/2}$ must vanish. Hence

$$1 + \beta = 0,$$

so

$$\boxed{\beta = -1}.$$

For this value,

$$y(t) = 2e^{-t/2} \longrightarrow 0 \qquad \text{as } t \to \infty.$$