Homework Assignment 3 – Solutions

MATH 2260 – Ordinary Differential Equations I

1. Let

$$f(y) = ay + by^2 = y(a + by), \qquad a > 0, \quad b > 0.$$

The critical points are

$$y = -\frac{a}{b} \qquad \text{and} \qquad y = 0.$$

Since $f(y)$ is an upward-opening parabola,

y	$(-\infty, -a/b)$	$-a/b$	$(-a/b, 0)$	0	$(0, \infty)$
$f(y)$	$+$	0	$-$	0	$+$

Thus the phase-line arrows point to the right for $y < -a/b$, to the left for $-a/b < y < 0$, and to the right for $y > 0$. Hence

$$\boxed{y = -\frac{a}{b} \text{ is asymptotically stable}}, \qquad \boxed{y = 0 \text{ is unstable}}.$$

For the specified initial values $y_0 \geq 0$, the relevant equilibrium is $y = 0$. If $y_0 = 0$, the solution remains at 0; if $y_0 > 0$, then $y'(t) > 0$, so the solution increases and moves away from 0.

2. Here

$$f(y) = y(y - 1)(y - 2).$$

The critical points are

$$y = 0, \qquad y = 1, \qquad y = 2.$$

The sign of f is

y	$(-\infty, 0)$	0	$(0, 1)$	1	$(1, 2)$	2	$(2, \infty)$
$f(y)$	$-$	0	$+$	0	$-$	0	$+$

Therefore

$$\boxed{y = 0 \text{ is unstable}}, \qquad \boxed{y = 1 \text{ is asymptotically stable}}, \qquad \boxed{y = 2 \text{ is unstable}}.$$

For $y_0 \geq 0$, the solution behavior is as follows: if $0 < y_0 < 1$, then the solution increases toward 1; if $1 < y_0 < 2$, it decreases toward 1; and if $y_0 > 2$, it increases away from 2. The values $y_0 = 0, 1, 2$ give the three equilibrium solutions.

3. The logistic equation is

$$\frac{dy}{dt} = ry\left(1 - \frac{y}{K}\right),$$

and, for $y(0) = y_0 > 0$, its solution is

$$y(t) = \frac{K}{1 + \left(\frac{K - y_0}{y_0}\right) e^{-rt}}.$$

(a) If $y_0 = K/3$, then

$$y(t) = \frac{K}{1 + 2e^{-rt}}.$$

The population has doubled when $y(\tau) = 2K/3$. Hence

$$\frac{2}{3} = \frac{1}{1 + 2e^{-r\tau}},$$

so

$$e^{-r\tau} = \frac{1}{4}.$$

Therefore

$$\boxed{\tau = \frac{\ln 4}{r}}.$$

For $r = 0.025$ per year,

$$\boxed{\tau = \frac{\ln 4}{0.025} \approx 55.45 \text{ years}}.$$

(b) If $y_0/K = \alpha$, then $y_0 = \alpha K$ and

$$\frac{y(t)}{K} = \frac{1}{1 + \left(\frac{1-\alpha}{\alpha}\right) e^{-rt}}.$$

Setting $y(T)/K = \beta$ gives

$$\beta = \frac{1}{1 + \left(\frac{1-\alpha}{\alpha}\right) e^{-rT}},$$

so

$$e^{-rT} = \frac{\alpha(1-\beta)}{\beta(1-\alpha)}.$$

Thus

$$\boxed{T = \frac{1}{r} \ln\left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right)}.$$

This formula shows that $T \to \infty$ as $\alpha \to 0^+$ or as $\beta \to 1^-$. For $r = 0.025$, $\alpha = 0.1$, and $\beta = 0.9$,

$$T = \frac{1}{0.025} \ln\left(\frac{(0.9)(0.9)}{(0.1)(0.1)}\right) = 40 \ln 81.$$

Hence

$$\boxed{T \approx 175.78 \text{ years}}.$$

4. Write the equation as

$$M(x, y) + N(x, y)y' = 0,$$

where

$$M(x, y) = 2x + 4y, \qquad N(x, y) = 2x - 2y.$$

Then

$$M_y = 4, \qquad N_x = 2.$$

Since $M_y \neq N_x$, the equation is not exact. Therefore

$$\boxed{\text{the differential equation is not exact}}.$$

5. Let

$$M(x,y) = 2xy^2 + 2y, \qquad N(x,y) = 2x^2y + 2x.$$

Then

$$M_y = 4xy + 2, \qquad N_x = 4xy + 2,$$

so the equation is exact. We seek ψ such that

$$\psi_x = M.$$

Integrating with respect to x,

$$\psi(x,y) = x^2y^2 + 2xy + h(y).$$

Then

$$\psi_y = 2x^2y + 2x + h'(y).$$

Comparing with N gives $h'(y) = 0$. Hence the implicit solution is

$$\boxed{x^2y^2 + 2xy = C}.$$

6. Here

$$M(x,y) = y, \qquad N(x,y) = 2xy - e^{-2y}.$$

The equation is not exact since

$$M_y = 1, \qquad N_x = 2y.$$

We compute

$$\frac{N_x - M_y}{M} = \frac{2y-1}{y} = 2 - \frac{1}{y},$$

which is a function of y only. Hence an integrating factor is

$$\mu(y) = \exp\left(\int \left(2 - \frac{1}{y}\right) dy\right) = \boxed{\frac{e^{2y}}{y}}, \qquad y \neq 0.$$

After multiplication by μ, the equation becomes

$$e^{2y}\,dx + \left(2xe^{2y} - \frac{1}{y}\right) dy = 0,$$

which is exact. Integrating the first term with respect to x gives

$$\psi(x,y) = xe^{2y} + h(y).$$

Since

$$\psi_y = 2xe^{2y} + h'(y) = 2xe^{2y} - \frac{1}{y},$$

we have $h'(y) = -1/y$, so $h(y) = -\ln|y|$. Therefore

$$\boxed{xe^{2y} - \ln|y| = C}, \qquad y \neq 0.$$

The original equation also has the constant solution

$$\boxed{y = 0}.$$

7. Let

$$M(x,y) = \frac{4x^3}{y^2} + \frac{3}{y}, \qquad N(x,y) = \frac{3x}{y^2} + 4y.$$

We find

$$M_y = -\frac{8x^3}{y^3} - \frac{3}{y^2}, \qquad N_x = \frac{3}{y^2}.$$

Then

$$\frac{N_x - M_y}{M} = \frac{\frac{8x^3}{y^3} + \frac{6}{y^2}}{\frac{4x^3}{y^2} + \frac{3}{y}} = \frac{2}{y}.$$

Thus an integrating factor depending only on y is

$$\mu(y) = \exp\left(\int \frac{2}{y}\,dy\right) = \boxed{y^2}.$$

Multiplying the equation by y^2 gives

$$(4x^3 + 3y)\,dx + (3x + 4y^3)\,dy = 0,$$

which is exact. A potential function is

$$\psi(x,y) = x^4 + 3xy + y^4.$$

Therefore the solution is

$$\boxed{x^4 + 3xy + y^4 = C}.$$

8. The characteristic equation is

$$r^2 + 3r + 2 = 0,$$

so

$$(r+1)(r+2) = 0.$$

Hence the roots are $r = -1$ and $r = -2$, and the general solution is

$$\boxed{y = c_1 e^{-t} + c_2 e^{-2t}}.$$

9. The characteristic equation is

$$r^2 - 2r - 2 = 0.$$

Its roots are

$$r = 1 \pm \sqrt{3}.$$

Therefore the general solution is

$$\boxed{y = c_1 e^{(1+\sqrt{3})t} + c_2 e^{(1-\sqrt{3})t}}.$$

10. The characteristic equation is

$$2r^2 - 3r + 1 = 0 = (2r-1)(r-1),$$

so the roots are $r = 1$ and $r = 1/2$. Thus

$$y = c_1 e^t + c_2 e^{t/2}.$$

The initial conditions give

$$c_1 + c_2 = 2, \qquad c_1 + \frac{1}{2} c_2 = \frac{1}{2}.$$

Solving,

$$c_1 = -1, \qquad c_2 = 3.$$

Hence

$$\boxed{y(t) = -e^t + 3e^{t/2}}.$$

To find the maximum, differentiate:

$$y'(t) = -e^t + \frac{3}{2} e^{t/2}.$$

Setting $y'(t) = 0$ gives

$$e^{t/2} = \frac{3}{2}, \qquad t = 2 \ln \frac{3}{2}.$$

At this point,

$$y_{\max} = -\left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) = \frac{9}{4}.$$

Therefore

$$\boxed{y_{\max} = \frac{9}{4} \quad \text{at} \quad t = 2 \ln \frac{3}{2}}.$$

Finally, $y(t) = 0$ when

$$-e^t + 3e^{t/2} = 0,$$

so $e^{t/2} = 3$ and hence

$$\boxed{t = 2 \ln 3}.$$

Thus the solution crosses the t-axis at

$$\boxed{(2 \ln 3, 0)}.$$