

Homework Assignment 2 – Solutions

MATH 2260 – Ordinary Differential Equations I

1. Solve

$$\frac{dy}{dx} = \frac{x^2}{y(1+x^3)}.$$

Separating the variables gives

$$y \, dy = \frac{x^2}{1+x^3} \, dx.$$

Integrating both sides,

$$\frac{y^2}{2} = \frac{1}{3} \ln |1+x^3| + C.$$

Therefore an implicit form of the general solution is

$$\boxed{y^2 = \frac{2}{3} \ln |1+x^3| + C},$$

on intervals where the differential equation is defined.

2. Solve

$$\frac{dy}{dx} = \cos^2 x \cos^2(2y).$$

For solutions for which $\cos(2y) \neq 0$, separate the variables:

$$\sec^2(2y) \, dy = \cos^2 x \, dx.$$

Hence

$$\frac{1}{2} \tan(2y) = \int \frac{1+\cos(2x)}{2} \, dx = \frac{x}{2} + \frac{1}{4} \sin(2x) + C.$$

Thus

$$\boxed{\tan(2y) = x + \frac{1}{2} \sin(2x) + C}.$$

There are also constant solutions for which $\cos(2y) = 0$, namely

$$\boxed{y = \frac{\pi}{4} + \frac{k\pi}{2}, \quad k \in \mathbb{Z}}.$$

3. Solve

$$\frac{dy}{dx} = \frac{x - e^{-x}}{y + e^y}.$$

Separating the variables gives

$$(y + e^y) \, dy = (x - e^{-x}) \, dx.$$

Integrating,

$$\frac{y^2}{2} + e^y = \frac{x^2}{2} + e^{-x} + C.$$

Equivalently,

$$\boxed{y^2 + 2e^y = x^2 + 2e^{-x} + C}.$$

4. Let $Q(t)$ denote the amount of salt in the tank, measured in grams. Since the volume remains 120 L, the salt enters at the rate

$$(\gamma \text{ g/L})(2 \text{ L/min}) = 2\gamma \text{ g/min},$$

and leaves at the rate

$$\frac{Q(t)}{120}(2) = \frac{Q(t)}{60} \text{ g/min}.$$

Thus

$$\frac{dQ}{dt} = 2\gamma - \frac{Q}{60}, \quad Q(0) = 0.$$

Rewriting,

$$Q' + \frac{1}{60}Q = 2\gamma.$$

The general solution is

$$Q(t) = 120\gamma + Ce^{-t/60}.$$

Using $Q(0) = 0$ gives $C = -120\gamma$. Hence

$$Q(t) = 120\gamma \left(1 - e^{-t/60}\right) \text{ g}.$$

Therefore

$$\lim_{t \rightarrow \infty} Q(t) = 120\gamma \text{ g}.$$

5. Consider

$$t(t-4)y' + y = 0, \quad y(2) = 1.$$

For $t \neq 0, 4$, write the equation in standard linear form:

$$y' + \frac{1}{t(t-4)}y = 0.$$

The coefficient

$$p(t) = \frac{1}{t(t-4)}$$

is continuous except at $t = 0$ and $t = 4$. The largest open interval containing the initial point $t = 2$ on which $p(t)$ is continuous is

$$0 < t < 4.$$

Thus the solution is certain to exist on $(0, 4)$.

6. Here

$$f(t, y) = \sqrt{1 - t^2 - y^2}.$$

The function f is real and continuous when

$$t^2 + y^2 \leq 1.$$

Also,

$$\frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1 - t^2 - y^2}},$$

which is continuous only when

$$t^2 + y^2 < 1.$$

Therefore the hypotheses of Theorem 2.4.2 are satisfied in the interior of the unit circle:

$$\boxed{t^2 + y^2 < 1}.$$

7. (a) Let

$$\phi(t) = e^{2t}.$$

Then

$$\phi'(t) = 2e^{2t},$$

so

$$\phi' - 2\phi = 2e^{2t} - 2e^{2t} = 0.$$

Hence $\phi(t) = e^{2t}$ is a solution of $y' - 2y = 0$.

Now let

$$y = c\phi(t) = ce^{2t}.$$

Then

$$y' = 2ce^{2t},$$

and therefore

$$y' - 2y = 2ce^{2t} - 2ce^{2t} = 0.$$

Thus

$$\boxed{y = ce^{2t}}$$

is a solution for every constant c .

(b) Let

$$\phi(t) = \frac{1}{t}, \quad t > 0.$$

Then

$$\phi'(t) = -\frac{1}{t^2},$$

and

$$\phi' + \phi^2 = -\frac{1}{t^2} + \frac{1}{t^2} = 0.$$

Thus $\phi(t) = 1/t$ is a solution of $y' + y^2 = 0$.

Now let

$$y = \frac{c}{t}.$$

Then

$$y' = -\frac{c}{t^2}, \quad y^2 = \frac{c^2}{t^2}.$$

Hence

$$y' + y^2 = \frac{c^2 - c}{t^2} = \frac{c(c-1)}{t^2}.$$

This is zero if and only if

$$c(c-1) = 0.$$

Therefore

$$\boxed{c = 0 \quad \text{or} \quad c = 1}.$$

Thus arbitrary constant multiples of a solution need not remain solutions for a nonlinear equation.

8. Since y_1 is a solution of

$$y' + p(t)y = 0,$$

we have

$$y_1' + p(t)y_1 = 0.$$

Since y_2 is a solution of

$$y' + p(t)y = g(t),$$

we have

$$y_2' + p(t)y_2 = g(t).$$

Let

$$y = y_1 + y_2.$$

Then

$$\begin{aligned} y' + p(t)y &= (y_1' + y_2') + p(t)(y_1 + y_2) \\ &= (y_1' + p(t)y_1) + (y_2' + p(t)y_2) \\ &= 0 + g(t) \\ &= g(t). \end{aligned}$$

Therefore

$$\boxed{y = y_1(t) + y_2(t)}$$

is also a solution of the second equation.