

Homework Assignment 1 – Solutions

MATH 2260 – Ordinary Differential Equations I

1. Consider

$$\frac{dy}{dt} = ay - b, \quad a \neq 0.$$

(a) An equilibrium solution is a constant solution $y = y_e$, so $dy/dt = 0$. Hence

$$0 = ay_e - b,$$

and therefore

$$y_e = \frac{b}{a}.$$

(b) Let

$$Y(t) = y(t) - y_e = y(t) - \frac{b}{a}.$$

Since y_e is constant,

$$Y' = y'.$$

Using the original differential equation,

$$Y' = ay - b = a \left(Y + \frac{b}{a} \right) - b = aY.$$

Thus Y satisfies

$$\frac{dY}{dt} = aY.$$

2. The population satisfies

$$\frac{dp}{dt} = 0.5p - 450 = \frac{1}{2}(p - 900).$$

The general solution is

$$p(t) = 900 + Ce^{t/2}.$$

If $p(0) = p_0$, then $C = p_0 - 900$, so

$$p(t) = 900 + (p_0 - 900)e^{t/2}.$$

(a) For $p(0) = 850$,

$$p(t) = 900 - 50e^{t/2}.$$

Extinction occurs when $p(t) = 0$, so

$$900 - 50e^{t/2} = 0, \quad e^{t/2} = 18.$$

Hence

$$t = 2 \ln 18 \approx 5.78 \text{ months}.$$

(b) If $0 < p_0 < 900$, extinction occurs when

$$0 = 900 + (p_0 - 900)e^{t/2}.$$

Thus

$$(900 - p_0)e^{t/2} = 900,$$

and therefore

$$t = 2 \ln \left(\frac{900}{900 - p_0} \right) \text{ months}.$$

(c) One year is 12 months. Setting the extinction time equal to 12 gives

$$12 = 2 \ln \left(\frac{900}{900 - p_0} \right).$$

Thus

$$e^6 = \frac{900}{900 - p_0},$$

so

$$900 - p_0 = 900e^{-6}.$$

Therefore

$$p_0 = 900(1 - e^{-6}) \approx 897.77.$$

3. The falling object satisfies

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \quad v(0) = 0.$$

The solution is

$$v(t) = 49 \left(1 - e^{-t/5} \right).$$

Hence the limiting velocity is

$$v_L = 49 \text{ m/s}.$$

(a) To reach 98% of the limiting velocity,

$$49 \left(1 - e^{-t/5} \right) = 0.98(49).$$

Therefore

$$e^{-t/5} = 0.02 = \frac{1}{50},$$

and hence

$$t = 5 \ln 50 \approx 19.56 \text{ s}.$$

(b) Let $x(t)$ be the distance fallen, with $x(0) = 0$. Since $x' = v$,

$$\frac{dx}{dt} = 49 \left(1 - e^{-t/5} \right).$$

Integrating,

$$x(t) = 49t + 245e^{-t/5} - 245.$$

At $t = 5 \ln 50$, we have $e^{-t/5} = 1/50$, so

$$\begin{aligned}x &= 49(5 \ln 50) + 245 \left(\frac{1}{50} \right) - 245 \\&= 245 \ln 50 - 240.1.\end{aligned}$$

Thus

$$\boxed{x \approx 718.35 \text{ m}}.$$

4. The differential equation is

$$t^2 \frac{d^2 y}{dt^2} + t \frac{dy}{dt} + 2y = \sin t.$$

The highest derivative is $d^2 y/dt^2$, so the equation is **second order**. Since y , y' , and y'' occur only to the first power and are not multiplied together, the equation is **linear**. Therefore,

$$\boxed{\text{second order and linear}}.$$

5. The differential equation is

$$\frac{dy}{dt} + ty^2 = 0.$$

The highest derivative is dy/dt , so the equation is **first order**. Because of the term y^2 , it is **nonlinear**. Thus

$$\boxed{\text{first order and nonlinear}}.$$

6. Suppose

$$y = e^{rt}.$$

Then

$$y' = re^{rt}, \quad y'' = r^2 e^{rt}.$$

Substitution into

$$y'' + y' - 6y = 0$$

gives

$$(r^2 + r - 6)e^{rt} = 0.$$

Since $e^{rt} \neq 0$,

$$r^2 + r - 6 = 0 = (r - 2)(r + 3).$$

Hence

$$\boxed{r = 2 \quad \text{or} \quad r = -3}.$$

7. Solve

$$y' - y = 2te^{2t}, \quad y(0) = 1.$$

This is a linear equation with integrating factor

$$\mu(t) = e^{\int -1 dt} = e^{-t}.$$

Multiplying by e^{-t} gives

$$(e^{-t}y)' = 2te^t.$$

Since

$$\int 2te^t dt = 2(t-1)e^t,$$

we obtain

$$e^{-t}y = 2(t-1)e^t + C,$$

or

$$y = 2(t-1)e^{2t} + Ce^t.$$

Using $y(0) = 1$,

$$1 = -2 + C,$$

so $C = 3$. Therefore

$$\boxed{y(t) = 2(t-1)e^{2t} + 3e^t}.$$

8. Solve

$$y' + \frac{2}{t}y = \frac{\cos t}{t^2}, \quad y(\pi) = 0, \quad t > 0.$$

The integrating factor is

$$\mu(t) = e^{\int 2/t dt} = t^2.$$

Multiplying the equation by t^2 gives

$$(t^2y)' = \cos t.$$

Integrating,

$$t^2y = \sin t + C.$$

Thus

$$y = \frac{\sin t + C}{t^2}.$$

The initial condition gives

$$0 = \frac{\sin \pi + C}{\pi^2},$$

so $C = 0$. Hence

$$\boxed{y(t) = \frac{\sin t}{t^2}, \quad t > 0}.$$

9. Solve

$$t^3y' + 4t^2y = e^{-t}, \quad y(-1) = 0, \quad t < 0.$$

Dividing by t^3 gives

$$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}.$$

For $t < 0$, an integrating factor is

$$\mu(t) = e^{\int 4/t dt} = t^4.$$

Multiplying by t^4 yields

$$(t^4y)' = te^{-t}.$$

Since

$$\int te^{-t} dt = -(t+1)e^{-t},$$

we obtain

$$t^4 y = -(t+1)e^{-t} + C.$$

Using $y(-1) = 0$ gives $C = 0$. Therefore

$$y(t) = -\frac{(t+1)e^{-t}}{t^4}, \quad t < 0.$$