

Chapter 3. Second Order Linear Equations

Section 3.1. Homogeneous Equations with Constant Coefficients (I)

Second order differential equations

A second order ordinary differential equation involves

$$t, \quad y(t), \quad y'(t), \quad y''(t).$$

A general form is

$$y'' = f(t, y, y').$$

Interpretation

If t is time, then

$$y(t) = \text{position}, \quad y'(t) = \text{velocity}, \quad y''(t) = \text{acceleration}.$$

Second order equations are therefore very natural in mechanical and physical models.

Linear second order equations

A second order equation is linear if it can be written as

$$y'' + p(t)y' + q(t)y = g(t).$$

Here p, q, g depend on t only. They do not depend on y or y' .

Equivalent form

Sometimes the equation is written as

$$P(t)y'' + Q(t)y' + R(t)y = G(t).$$

If $P(t) \neq 0$, divide by $P(t)$ to get

$$p(t) = \frac{Q(t)}{P(t)}, \quad q(t) = \frac{R(t)}{P(t)}, \quad g(t) = \frac{G(t)}{P(t)}.$$

Linear or nonlinear?

Linear examples

$$y'' + 3y' - 4y = \sin t$$

$$t^2 y'' + ty' + (1+t)y = e^{-t}$$

$$y'' + p(t)y' + q(t)y = 0$$

Only y, y', y'' occur to the first power.

Nonlinear examples

$$y'' + yy' + y = 0$$

$$y'' + (y')^2 = \cos t$$

$$y'' + \sin y = 0$$

Here the unknown function enters nonlinearly.

In this lecture we focus on linear homogeneous equations with constant coefficients.

Homogeneous equations

The equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is called **homogeneous** if

$$g(t) = 0.$$

Thus a homogeneous linear equation has the form

$$y'' + p(t)y' + q(t)y = 0.$$

Constant coefficient case

In Section 3.1, we study

$$ay'' + by' + cy = 0, \quad a \neq 0,$$

where a, b, c are constants.

A motivating example

Solve

$$y'' - y = 0.$$

This asks for a function whose second derivative is the same as the original function.

Two familiar solutions

$$y_1(t) = e^t, \quad y_2(t) = e^{-t}.$$

Indeed,

$$(e^t)'' = e^t, \quad (e^{-t})'' = e^{-t}.$$

Constant multiples are still solutions

If $y(t)$ solves

$$y'' - y = 0,$$

then any constant multiple $cy(t)$ also solves it because

$$(cy)'' - cy = c(y'' - y) = 0.$$

Therefore

$$c_1 e^t, \quad c_2 e^{-t}$$

are solutions for any constants c_1, c_2 .

Sums are also solutions

For the equation

$$y'' - y = 0,$$

consider

$$y = c_1 e^t + c_2 e^{-t}.$$

Then

$$y' = c_1 e^t - c_2 e^{-t},$$

$$y'' = c_1 e^t + c_2 e^{-t} = y.$$

Thus, $y'' - y = 0$.

Family of solutions

$$y = c_1 e^t + c_2 e^{-t}$$

is a two-parameter family of solutions.

Example 1: impose initial conditions

Solve

$$y'' - y = 0, \quad y(0) = 2, \quad y'(0) = -1.$$

From the previous slide,

$$y = c_1 e^t + c_2 e^{-t}.$$

Then

$$y' = c_1 e^t - c_2 e^{-t}.$$

Apply $y(0) = 2$:

$$c_1 + c_2 = 2.$$

Apply $y'(0) = -1$:

$$c_1 - c_2 = -1.$$

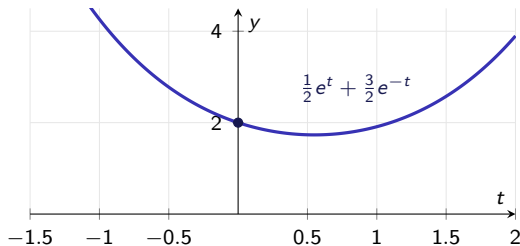
Solving,

$$c_1 = \frac{1}{2}, \quad c_2 = \frac{3}{2}.$$

Example 1: final answer and graph

Therefore

$$y(t) = \frac{1}{2}e^t + \frac{3}{2}e^{-t}.$$



From the example to the general method

For

$$ay'' + by' + cy = 0,$$

the previous example suggests trying exponential functions.

Trial solution

Assume

$$y = e^{rt}$$

for some number r to be determined. Then

$$y' = re^{rt}, \quad y'' = r^2 e^{rt}.$$

Substitution gives

$$ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0.$$

The characteristic equation

Since $e^{rt} \neq 0$, the equation

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

reduces to

$$ar^2 + br + c = 0.$$

Definition

The algebraic equation

$$ar^2 + br + c = 0$$

is called the **characteristic equation** of

$$ay'' + by' + cy = 0.$$

Each root r gives an exponential solution e^{rt} .

Three possible root cases

The characteristic equation

$$ar^2 + br + c = 0$$

is quadratic. Its roots are determined by the discriminant

$$\Delta = b^2 - 4ac.$$

Cases

1. $\Delta > 0$: two real distinct roots $r_1 \neq r_2$.
2. $\Delta = 0$: one repeated real root.
3. $\Delta < 0$: complex conjugate roots.

In Section 3.1 we focus on Case 1: two real distinct roots.

Real distinct roots

Suppose

$$ar^2 + br + c = 0$$

has two real distinct roots

$$r_1 \neq r_2.$$

Then

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$$

are solutions of

$$ay'' + by' + cy = 0.$$

By linearity,

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is also a solution.

Why the linear combination works

Let

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

Then

$$y' = c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t},$$

$$y'' = c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}.$$

Substitute into $ay'' + by' + cy$:

$$\begin{aligned} ay'' + by' + cy &= c_1(ar_1^2 + br_1 + c)e^{r_1 t} \\ &\quad + c_2(ar_2^2 + br_2 + c)e^{r_2 t} = 0. \end{aligned}$$

Both parentheses are zero because r_1, r_2 are roots.

Example 2: general solution

Find the general solution of

$$y'' + 5y' + 6y = 0.$$

The characteristic equation is

$$r^2 + 5r + 6 = 0.$$

Factor:

$$r^2 + 5r + 6 = (r + 2)(r + 3).$$

Thus

$$r_1 = -2, \quad r_2 = -3.$$

Therefore

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}.$$