

Chapter 2. First Order Differential Equations

Section 6. Exact Equations and Integrating Factors

The hidden function $\psi(x, y)$

Let

$$M(x, y) + N(x, y)y' = 0.$$

Equivalently,

$$M(x, y) dx + N(x, y) dy = 0.$$

Exact equation

The equation is exact if there is a function $\psi(x, y)$ such that

$$\psi_x(x, y) = M(x, y), \quad \psi_y(x, y) = N(x, y).$$

Then, along a solution $y = y(x)$,

$$\frac{d}{dx}\psi(x, y(x)) = \psi_x + \psi_y y' = M + Ny' = 0.$$

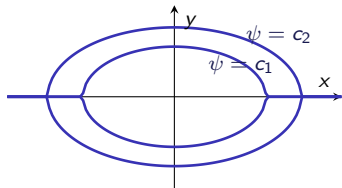
Solutions are level curves

If $M dx + N dy = 0$ is exact, then the solution is given implicitly by

$$\psi(x, y) = c.$$

The solution curves are the level curves of the function ψ .

- ▶ Different constants c give different integral curves.
- ▶ An initial condition determines the constant c .



The exactness test

Suppose M, N, M_y, N_x are continuous in a suitable region. Then

$$M(x, y) + N(x, y)y' = 0$$

is exact if and only if

$$M_y(x, y) = N_x(x, y).$$

Why this condition is natural

If $M = \psi_x$ and $N = \psi_y$, then

$$M_y = \psi_{xy}, \quad N_x = \psi_{yx}.$$

For continuously differentiable functions, mixed partial derivatives agree:

$$\psi_{xy} = \psi_{yx}.$$

Algorithm for exact equations

Given

$$M(x, y) + N(x, y)y' = 0.$$

Step-by-step method

1. Check exactness: compute M_y and N_x .
2. If $M_y = N_x$, find ψ from

$$\psi_x = M.$$

That is,

$$\psi(x, y) = \int M(x, y) dx + h(y).$$

3. Differentiate with respect to y and set $\psi_y = N$.
4. Solve for $h'(y)$, integrate to find $h(y)$.
5. Write the solution as $\psi(x, y) = c$.

Example 1: a quick exact equation

Solve

$$2x + y^2 + 2xy y' = 0.$$

Here

$$M(x, y) = 2x + y^2, \quad N(x, y) = 2xy.$$

Check exactness:

$$M_y = 2y, \quad N_x = 2y.$$

Thus the equation is exact.

Find ψ

$$\psi_x = 2x + y^2 \implies \psi = x^2 + xy^2 + h(y).$$

Then

$$\psi_y = 2xy + h'(y) = N = 2xy,$$

so $h'(y) = 0$. Hence

$$\boxed{x^2 + xy^2 = c.}$$

Using an initial condition

For the exact equation

$$2x + y^2 + 2xy y' = 0,$$

we found

$$\psi(x, y) = x^2 + xy^2.$$

If the initial condition is $y(1) = 2$, then

$$c = \psi(1, 2) = 1^2 + 1 \cdot 2^2 = 5.$$

Therefore the particular solution is

$$x^2 + xy^2 = 5.$$

Remark

Exact equations often give implicit solutions. Solving explicitly for y may be possible, but it is not always necessary.

Example 2: exact equation

Solve

$$(y \cos x + 2xe^y) + (\sin x + x^2e^y - 1)y' = 0.$$

Here

$$M = y \cos x + 2xe^y, \quad N = \sin x + x^2e^y - 1.$$

Check:

$$M_y = \cos x + 2xe^y, \quad N_x = \cos x + 2xe^y.$$

So the equation is exact.

Find ψ from $\psi_x = M$

$$\psi = \int (y \cos x + 2xe^y) dx = y \sin x + x^2e^y + h(y).$$

Example 2: finish the solution

We have

$$\psi(x, y) = y \sin x + x^2 e^y + h(y).$$

Differentiate with respect to y :

$$\psi_y = \sin x + x^2 e^y + h'(y).$$

Set $\psi_y = N$:

$$\sin x + x^2 e^y + h'(y) = \sin x + x^2 e^y - 1.$$

Thus

$$h'(y) = -1, \quad h(y) = -y.$$

Therefore

$$\psi(x, y) = y \sin x + x^2 e^y - y.$$

The implicit solution is

$$y \sin x + x^2 e^y - y = c.$$

When the equation is not exact

Consider

$$(3xy + y^2) + (x^2 + xy)y' = 0.$$

Here

$$M = 3xy + y^2, \quad N = x^2 + xy.$$

Then

$$M_y = 3x + 2y, \quad N_x = 2x + y.$$

Since

$$M_y \neq N_x,$$

the equation is not exact.

Next idea

Try to multiply the whole equation by a nonzero function $\mu(x, y)$ so that it becomes exact.

Integrating factors

Start with

$$M(x, y) dx + N(x, y) dy = 0.$$

Multiply by an integrating factor μ :

$$\mu M dx + \mu N dy = 0.$$

For the new equation to be exact, we need

$$(\mu M)_y = (\mu N)_x.$$

General difficulty

This condition is a partial differential equation for μ :

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0.$$

In general this is hard to solve, so we look for special cases.

Integrating factor depending only on x

Suppose $\mu = \mu(x)$ only. Then

$$(\mu M)_y = \mu M_y, \quad (\mu N)_x = \mu N_x + N \frac{d\mu}{dx}.$$

Exactness gives

$$\frac{d\mu}{dx} = \frac{M_y - N_x}{N} \mu.$$

Useful test

If

$$\frac{M_y - N_x}{N} = P(x)$$

is a function of x only, then

$$\mu(x) = \exp \left(\int P(x) dx \right).$$

Example 3: find an integrating factor

For

$$(3xy + y^2) + (x^2 + xy)y' = 0,$$

Then we have

$$M_y = 3x + 2y, \quad N_x = 2x + y.$$

Now

$$\frac{M_y - N_x}{N} = \frac{x + y}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x}.$$

This depends only on x . Therefore

$$\mu(x) = \exp\left(\int \frac{1}{x} dx\right) = x.$$

Multiplying the equation by x gives

$$(3x^2y + xy^2) + (x^3 + x^2y)y' = 0.$$

Example 3: solve after multiplying

For the multiplied equation,

$$\tilde{M} = 3x^2y + xy^2, \quad \tilde{N} = x^3 + x^2y.$$

It is exact because

$$\tilde{M}_y = 3x^2 + 2xy = \tilde{N}_x.$$

Find ψ :

$$\psi_x = 3x^2y + xy^2.$$

Thus

$$\psi = x^3y + \frac{1}{2}x^2y^2 + h(y).$$

Then

$$\psi_y = x^3 + x^2y + h'(y) = \tilde{N} = x^3 + x^2y,$$

so $h'(y) = 0$. Hence

$$x^3y + \frac{1}{2}x^2y^2 = c.$$

Integrating factor depending only on y

A similar test works for $\mu = \mu(y)$.

Useful test

If

$$\frac{N_x - M_y}{M} = Q(y)$$

is a function of y only, then

$$\mu(y) = \exp \left(\int Q(y) dy \right).$$

Summary

1. $M dx + N dy = 0$ is exact when $M = \psi_x$ and $N = \psi_y$.

2. Exactness test:

$$M_y = N_x.$$

3. Solution of an exact equation:

$$\psi(x, y) = c.$$

4. If the equation is not exact, try an integrating factor.

5. Useful special cases:

$$\frac{M_y - N_x}{N} = P(x), \quad \frac{N_x - M_y}{M} = Q(y).$$