

Chapter 2. First Order Differential Equations

Section 5. Autonomous Equations and Population Dynamics

Autonomous equations

An autonomous first order equation has the form

$$\frac{dy}{dt} = f(y).$$

The independent variable t does not appear explicitly.

Equilibrium solutions

A constant solution $y(t) = y_e$ satisfies

$$f(y_e) = 0.$$

The roots of $f(y) = 0$ are called **critical points** or **equilibrium points**.

- ▶ If $f(y) > 0$, solutions increase.
- ▶ If $f(y) < 0$, solutions decrease.

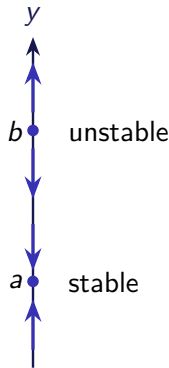
Phase line: the basic geometric tool

For $y' = f(y)$:

- ▶ mark the critical points on the y -axis;
- ▶ use arrows to indicate whether y increases or decreases;
- ▶ read the long-time behavior from the arrows.

Stability language

- ▶ arrows point toward y_e : asymptotically stable;
- ▶ arrows point away from y_e : unstable.



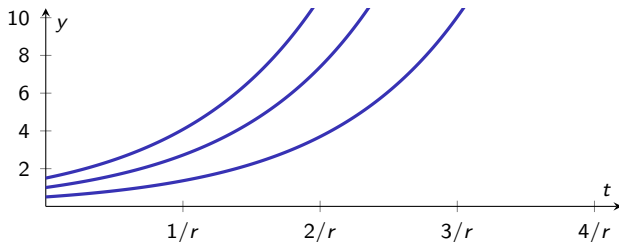
Exponential growth

The simplest population model assumes that the growth rate is proportional to the current population:

$$\boxed{\frac{dy}{dt} = ry}, \quad r > 0.$$

With $y(0) = y_0$,

$$\boxed{y(t) = y_0 e^{rt}}.$$



Logistic growth: modifying the growth rate

Instead of a constant growth rate r , let the per-capita growth rate depend on the population:

$$\frac{dy}{dt} = h(y)y.$$

Choose the simplest decreasing linear function

$$h(y) = r - ay, \quad r > 0, \quad a > 0.$$

Then

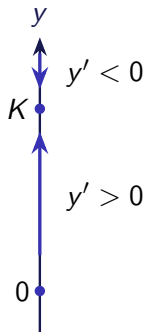
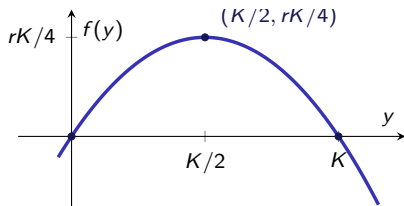
$$\frac{dy}{dt} = (r - ay)y = r \left(1 - \frac{y}{K}\right) y, \quad K = \frac{r}{a}.$$

Interpretation

r is the intrinsic growth rate, and K is the carrying capacity or saturation level.

Logistic equation: graph of $f(y)$ and phase line

$$y' = f(y) = r \left(1 - \frac{y}{K} \right) y.$$

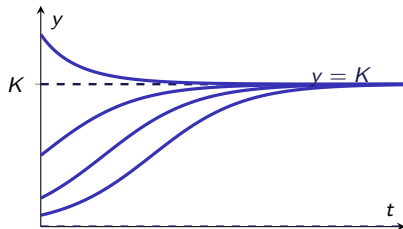


Logistic growth: solution curves

The equilibrium solutions are

$$y = 0, \quad y = K.$$

- ▶ $0 < y_0 < K$: population increases to K .
- ▶ $y_0 > K$: population decreases to K .
- ▶ $y = 0$ is unstable.
- ▶ $y = K$ is asymptotically stable.



Solving the logistic equation

$$\frac{dy}{dt} = r \left(1 - \frac{y}{K} \right) y, \quad y(0) = y_0.$$

Separate variables and use partial fractions:

$$\frac{dy}{y(1 - y/K)} = r dt, \quad \frac{1}{y(1 - y/K)} = \frac{1}{y} + \frac{1/K}{1 - y/K}.$$

Integrate:

$$\ln |y| - \ln \left| 1 - \frac{y}{K} \right| = rt + C.$$

Thus

$$\frac{y}{1 - y/K} = Ae^{rt}.$$

Apply $y(0) = y_0$:

$$A = \frac{y_0}{1 - y_0/K} = \frac{Ky_0}{K - y_0}.$$

$$\text{Since } \frac{y}{1 - y/K} = \frac{Ky}{K - y},$$

$$\frac{Ky}{K - y} = \frac{Ky_0}{K - y_0} e^{rt}.$$

Then

$$y(K - y_0) = y_0(K - y)e^{rt},$$

so

$$y(K - y_0 + y_0 e^{rt}) = Ky_0 e^{rt}.$$

Therefore

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}.$$

Halibut population

The logistic model is used for the biomass $y(t)$ of halibut in the Pacific Ocean:

$$r = 0.71/\text{year}, \quad K = 80.5 \times 10^6 \text{ kg}, \quad y_0 = 0.25K.$$

Scale by K :

$$\frac{y(t)}{K} = \frac{y_0/K}{(y_0/K) + [1 - (y_0/K)]e^{-rt}}.$$

Then

$$\frac{y(2)}{K} = \frac{0.25}{0.25 + 0.75e^{-1.42}} \approx 0.5797,$$

so

$$y(2) \approx 46.7 \times 10^6 \text{ kg}.$$

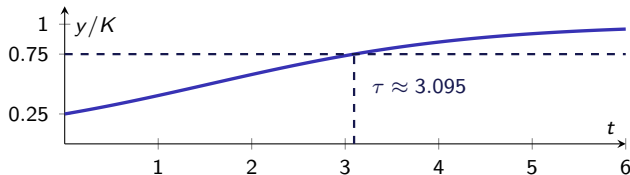
Halibut example: reaching $0.75K$

To solve for the time at which y/K reaches a prescribed level, solve the logistic formula for t :

$$t = -\frac{1}{r} \ln \left(\frac{(y_0/K)[1 - y/K]}{(y/K)[1 - y_0/K]} \right).$$

With $y_0/K = 0.25$, $y/K = 0.75$, and $r = 0.71$,

$$\tau = -\frac{1}{0.71} \ln \left(\frac{(0.25)(0.25)}{(0.75)(0.75)} \right) = \frac{1}{0.71} \ln 9 \approx 3.095 \text{ years.}$$



Logistic growth with a threshold

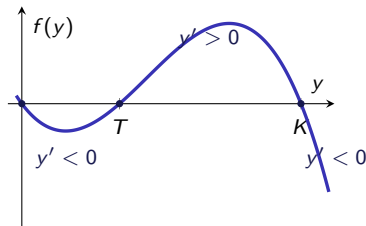
To prevent unlimited growth above the threshold, introduce a carrying capacity K :

$$y' = -r \left(1 - \frac{y}{T}\right) \left(1 - \frac{y}{K}\right) y, \quad 0 < T < K.$$

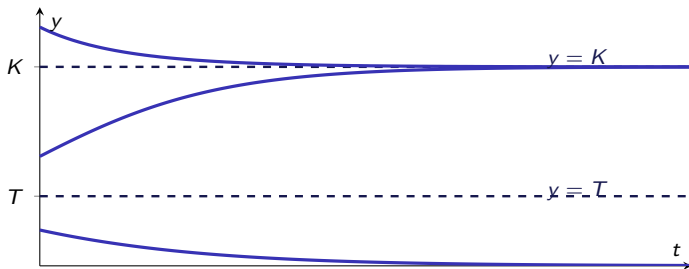
Critical points:

$$y = 0, \quad y = T, \quad y = K.$$

- ▶ $y = 0$ is asymptotically stable.
- ▶ $y = T$ is unstable.
- ▶ $y = K$ is asymptotically stable.



Logistic threshold model: solution behavior



- ▶ If $0 < y_0 < T$, the population declines toward extinction.
- ▶ If $T < y_0 < K$, the population increases toward the carrying capacity.
- ▶ If $y_0 > K$, the population decreases toward the carrying capacity.

Summary

1. For $y' = f(y)$, equilibria are roots of $f(y) = 0$.
2. A phase line often gives the long-time behavior without solving the ODE.
3. Stability is determined by the arrows near each equilibrium.
4. Logistic growth has carrying capacity K .
5. Threshold models explain extinction below a critical population level.

For practice

For any autonomous equation, first sketch $f(y)$ to build the phase line, then sketch representative solution curves in the ty -plane.