

Chapter 2. First Order Differential Equations

Section 4. Differences Between Linear and Nonlinear Equations

Linear equations: existence and uniqueness

Consider the linear initial value problem

$$y' + p(t)y = g(t), \quad y(t_0) = y_0.$$

Theorem 2.4.1

If p and g are continuous on an open interval

$$I : \alpha < t < \beta$$

containing t_0 , then there is a unique solution $y = \phi(t)$ on all of I .

Linear equations: explicit formula

Choose

$$\mu(t) = \exp \left(\int_{t_0}^t p(s) ds \right), \quad \mu(t_0) = 1.$$

Then

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)g(t).$$

Hence the solution of

$$y' + p(t)y = g(t), \quad y(t_0) = y_0$$

is

$$y(t) = \frac{1}{\mu(t)} \left(\int_{t_0}^t \mu(s)g(s) ds + y_0 \right).$$

Important point

This implies the existence and uniqueness (why?).

Nonlinear equations: local existence and uniqueness

For a nonlinear problem, write

$$y' = f(t, y), \quad y(t_0) = y_0.$$

Theorem 2.4.2

If f and $\partial f / \partial y$ are continuous in a rectangle

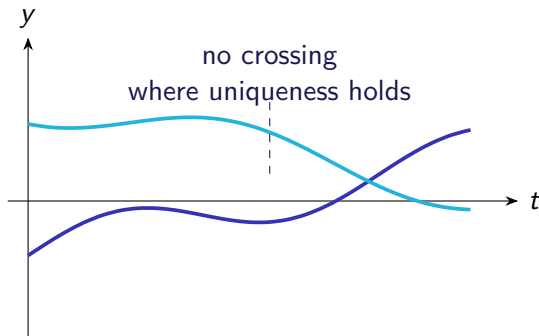
$$\alpha < t < \beta, \quad \gamma < y < \delta$$

containing (t_0, y_0) , then there is a unique solution in some interval

$$t_0 - h < t < t_0 + h.$$

Geometric meaning of uniqueness

If uniqueness holds, then two solution curves cannot cross.



Reason

If two graphs intersect at (t_*, y_*) , then both satisfy the same initial condition $y(t_*) = y_*$.

Example 1: a linear equation

Consider

$$ty' + 2y = 4t^2, \quad y(1) = 2.$$

Put it in standard linear form:

$$y' + \frac{2}{t}y = 4t.$$

Here

$$p(t) = \frac{2}{t}, \quad g(t) = 4t.$$

Note that p is continuous on $t > 0$ and on $t < 0$, but not at $t = 0$. Since $t_0 = 1$, we can use Theorem 2.4.1 to conclude that the solution is unique on

$$0 < t < \infty.$$

Indeed,

$$y = t^2 + \frac{1}{t^2}, \quad t > 0.$$

Example 2: a nonlinear equation

Consider

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}, \quad y(0) = -1.$$

Here

$$f(x, y) = \frac{3x^2 + 4x + 2}{2(y - 1)}, \quad \frac{\partial f}{\partial y} = -\frac{3x^2 + 4x + 2}{2(y - 1)^2}.$$

Both are continuous except on the line

$$y = 1.$$

Since $(0, -1)$ is away from $y = 1$, Theorem 2.4.2 guarantees a unique solution near $x = 0$. But solving shows that this particular solution exists for

$$x > -2.$$

When the theorem does not apply

For the same differential equation,

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)},$$

change the initial condition to $y(0) = 1$. Now the initial point lies on the line $y = 1$, where f is not continuous.

The theorem gives no conclusion

It does not say uniqueness fails; it only says the theorem cannot be used.

Separating variables gives $y^2 - 2y = x^3 + 2x^2 + 2x + c$. With $y(0) = 1$, $c = -1$, and hence

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x}.$$

So two solution branches pass through the same initial point for $x > 0$.

Example 3: nonuniqueness

Consider

$$y' = y^{1/3}, \quad y(0) = 0, \quad t \geq 0.$$

The function $f(t, y) = y^{1/3}$ is continuous, but

$$\frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3}$$

is not continuous at $y = 0$.

Several solutions

$$y = 0, \quad y = \left(\frac{2t}{3}\right)^{3/2}, \quad y = -\left(\frac{2t}{3}\right)^{3/2}$$

all satisfy the same initial value problem.

Interval of definition: nonlinear blow-up

Consider

$$y' = y^2, \quad y(0) = 1.$$

Theorem 2.4.2 applies because $f(t, y) = y^2$ and $f_y = 2y$ are continuous everywhere. Separating variables gives

$$y^{-2} dy = dt, \quad -\frac{1}{y} = t + c.$$

Using $y(0) = 1$ gives

$$y = \frac{1}{1 - t}.$$

Key point

The solution becomes unbounded as $t \rightarrow 1^-$. Thus its interval is

$$-\infty < t < 1.$$

General solution: linear vs nonlinear

For a first order linear equation, there is a solution containing one arbitrary constant, and all solutions are obtained from it.

For nonlinear equations, however, a formula with one arbitrary constant may miss solutions.

Example

For $y' = y^2$,

$$y = -\frac{1}{t + c}$$

is a family of nonzero solutions, but the solution

$$\boxed{y(t) = 0}$$

is not obtained from this formula.

Implicit solutions

For nonlinear equations, an explicit formula is often unavailable. Frequently we can only find an equation

$$F(t, y) = 0$$

that defines the solution implicitly.

Interpretation

The graph of $F(t, y) = 0$ is an integral curve. For each value of t , one may need to solve this equation for y .

Practical point

If solving for y is difficult, use numerical computation and graph the resulting points.

Summary: linear and nonlinear equations

Linear equations

- ▶ Existence and uniqueness on the whole interval where p and g are continuous.
- ▶ Explicit solution formula using an integrating factor.
- ▶ Singularities occur only where coefficients are singular.

Nonlinear equations

- ▶ Existence and uniqueness are usually local.
- ▶ Solutions may fail to be unique if hypotheses break down.
- ▶ Solutions may blow up at points depending on the initial condition.
- ▶ Exact solutions are often implicit or unavailable.