

Chapter 2. First Order Differential Equations

Section 3. Modeling with First Order Equations

Mathematical modeling

Three steps

1. **Construct the model:** variables, units, assumptions, and physical law.
2. **Analyze the model:** solve or approximate the differential equation.
3. **Interpret the result:** check units, limiting behavior, and reasonableness.

A central modeling principle

Many first order models in this section come from a balance law:

$$\text{rate of change} = \text{rate in} - \text{rate out.}$$

For other models, the governing law may be Newton's law:

$$m \frac{dv}{dt} = \text{net force.}$$

Comment:

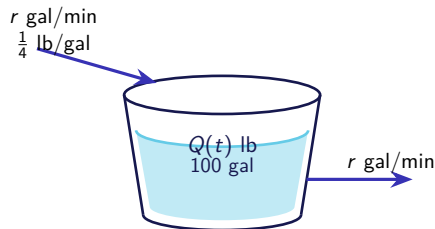
Every term in a differential equation must have the same physical units. This is a useful way to catch modeling mistakes.

Example 1: mixing in a tank

A tank contains 100 gal of water and Q_0 lb of salt initially.

- ▶ Inflow concentration: $\frac{1}{4}$ lb/gal.
- ▶ Inflow and outflow rate: r gal/min.
- ▶ The mixture is well stirred.

Let $Q(t)$ be the salt amount in lb.



Example 1: build the differential equation

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}.$$

Inflow of salt:

$$\left(\frac{1}{4}\right) r = \frac{r}{4} \text{ lb/min.}$$

Outflow concentration is $Q(t)/100$ lb/gal, so

$$\text{rate out} = r \frac{Q(t)}{100} = \frac{rQ}{100} \text{ lb/min.}$$

Therefore

$$\boxed{\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}}, \quad \boxed{Q(0) = Q_0.}$$

Example 1: solve and interpret

Standard form:

$$Q' + \frac{r}{100}Q = \frac{r}{4}.$$

The equilibrium, or limiting amount, is found from $Q' = 0$:

$$Q_L = 25 \text{ lb.}$$

Solving gives

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100}.$$

Example 1: time estimate and graph

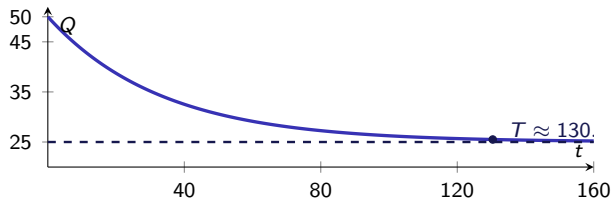
If $r = 3$ and $Q_0 = 2Q_L = 50$, then

$$Q(t) = 25 + 25e^{-0.03t}.$$

Find the time T at which $Q(t)$ has the value 25.5:

$$25e^{-0.03T} = 0.5,$$

$$T = \frac{\ln 50}{0.03} \approx 130.4 \text{ min.}$$



Example 2: chemical in a pond

A pond contains 10^7 gal of fresh water.

► Inflow/outflow: 5×10^6 gal/year.

► Incoming concentration: $\gamma(t) = 2 + \sin 2t$ g/gal.

Let $Q(t)$ be the chemical amount in grams. Then

$$\frac{dQ}{dt} = (5 \times 10^6)(2 + \sin 2t) - \frac{Q}{2}.$$

Using $q = Q/10^6$ gives

$$\boxed{q' + \frac{1}{2}q = 10 + 5 \sin 2t}, \quad \boxed{q(0) = 0}.$$

Example 2: solve the pond model

The integrating factor is

$$\mu(t) = e^{t/2}.$$

The general solution is

$$q(t) = 20 - \frac{40}{17} \cos 2t + \frac{10}{17} \sin 2t + Ce^{-t/2}.$$

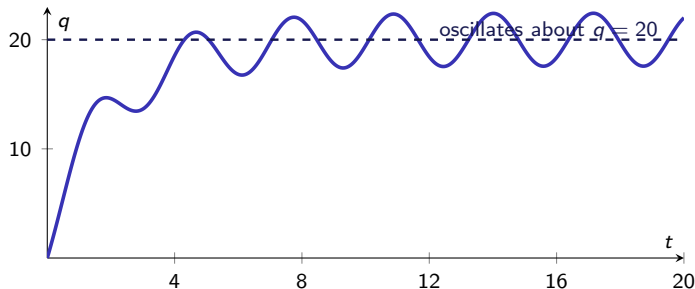
Using $q(0) = 0$ gives $C = -300/17$, so

$$q(t) = 20 - \frac{40}{17} \cos 2t + \frac{10}{17} \sin 2t - \frac{300}{17} e^{-t/2}.$$

Long-term behavior

The exponential term disappears, but the periodic terms remain.

Example 2: transient plus periodic steady state



- ▶ $-\frac{300}{17}e^{-t/2}$ is the transient term.
- ▶ The sine and cosine terms persist because the inflow concentration is periodic.

Example 2: model assumptions

The pond model assumes:

- no water is lost by evaporation or seepage;
- no water is gained by rainfall;
- the chemical is neither absorbed nor destroyed;
- the concentration is uniform throughout the pond.

Comment:

An equation may be solved exactly, but its physical accuracy depends on the assumptions.

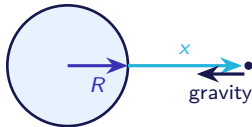
Example 3: escape velocity

Let x be altitude above the earth and R the radius of the earth. The gravitational force is

$$w(x) = -\frac{mgR^2}{(R+x)^2}.$$

Newton's law gives

$$m \frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \quad v(0) = v_0.$$



Example 3: use position as independent variable

Because the equation contains both x and v , use the chain rule:

$$\frac{dv}{dt} = \frac{dv}{dx} \frac{dx}{dt} = v \frac{dv}{dx}.$$

Then the equation becomes

$$v \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}.$$

Separate and integrate:

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + C.$$

Using $v = v_0$ at $x = 0$ gives

$$v = \pm \sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+x}}.$$

Example 3: maximum altitude and escape velocity

At maximum altitude $x = \xi$, the velocity is zero. Therefore

$$0 = v_0^2 - 2gR + \frac{2gR^2}{R + \xi}.$$

Solving,

$$\xi = \frac{v_0^2 R}{2gR - v_0^2},$$

$$v_0 = \sqrt{\frac{2gR\xi}{R + \xi}}.$$

Letting $\xi \rightarrow \infty$ gives the escape velocity

$$v_e = \sqrt{2gR}.$$