Chapter 2. First Order Differential Equations

Section 2. Separable Equations

The class of equations

Separable form

$$\boxed{\frac{dy}{dx} = g(x)h(y)}$$

- Separate all y-terms with dy and all x-terms with dx.
- Integrate both sides.
- The result is often an implicit relation between x and y.

Equivalent differential form

A separable equation may also be written as

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

Multiplying by dx gives

$$\boxed{M(x)\,dx + N(y)\,dy = 0.}$$

Then integrate each part:

$$\boxed{\int M(x)\,dx + \int N(y)\,dy = C.}$$

Comment:

This form is useful when the equation is not written explicitly as $y' = f(x, y)$.

Basic procedure for an IVP

For

$$\frac{dy}{dx} = g(x)h(y), \qquad y(x_0) = y_0,$$

first check constant solutions from $h(y) = 0$. For nonconstant solutions,

$$\frac{dy}{h(y)} = g(x)\,dx.$$

The IVP can be represented implicitly as

$$\boxed{\int_{y_0}^{y} \frac{ds}{h(s)} = \int_{x_0}^{x} g(s)\,ds.}$$

Warning: lost equilibrium solutions

If we divide by $h(y)$, we may lose constant solutions satisfying

$$h(y) = 0.$$

Therefore:

First find constant equilibrium solutions.

Then separate variables for the remaining solutions.

Finally use the initial condition and determine the interval of validity.

Example 1: separate and integrate

Show that

$$\frac{dy}{dx} = \frac{x^2}{1-y^2}$$

is separable. Rewrite it as

$$-x^2 + (1-y^2)\frac{dy}{dx} = 0,$$

or

$$-x^2\,dx + (1-y^2)\,dy = 0.$$

Integrating gives

$$-\frac{x^3}{3} + y - \frac{y^3}{3} = C.$$

stealth- Equivalently,

$$\boxed{-x^3 + 3y - y^3 = C.}$$

General implicit solution

Suppose

$$M(x) + N(y)y' = 0.$$

Let $H_1'(x) = M(x)$ and $H_2'(y) = N(y)$. Then

$$H_1'(x) + H_2'(y)y' = 0$$

means

$$\frac{d}{dx}\left[H_1(x) + H_2(y)\right] = 0.$$

Therefore

$$\boxed{H_1(x) + H_2(y) = C.}$$

This is usually the most natural final form for a nonlinear separable equation.

Example 2: set up the IVP

Solve

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad y(0) = -1.$$

Separate variables:

$$2(y-1)\,dy = (3x^2 + 4x + 2)\,dx.$$

Integrate:

$$y^2 - 2y = x^3 + 2x^2 + 2x + C.$$

Using $y(0) = -1$ gives $C = 3$, so

$$\boxed{y^2 - 2y = x^3 + 2x^2 + 2x + 3.}$$

Example 2: explicit solution and interval

Solve the quadratic for y:

$$y = 1 \pm \sqrt{x^3 + 2x^2 + 2x + 4}.$$

The initial condition $y(0) = -1$ selects the minus sign:

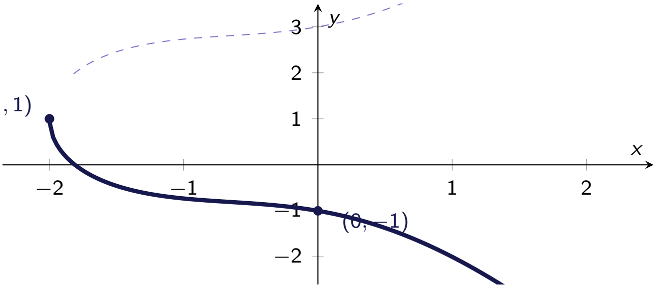
$$\boxed{y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}.}$$

The expression under the radical has its real zero at $x = -2$, so the maximal interval containing 0 is

$$\boxed{x > -2.}$$

At $(-2, 1)$ the tangent line is vertical.

Example 2: graph



Dashed curves show the two branches. The heavy branch is the solution satisfying the initial condition.

Example 3: implicit solution

Solve

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}.$$

Separate variables:

$$(4 + y^3)\, dy = (4x - x^3)\, dx.$$

Integrate and multiply by 4:

$$16y + y^4 = 8x^2 - x^4 + C.$$

Thus the integral curves are

$$\boxed{y^4 + 16y + x^4 - 8x^2 = C.}$$

Example 3: IVP and vertical tangents

For the solution through $(0, 1)$,

$$C = 1 + 16 = 17,$$

so

$$\boxed{y^4 + 16y + x^4 - 8x^2 = 17.}$$

Vertical tangents occur when

$$4 + y^3 = 0, \qquad y = -\sqrt[3]{4} \approx -1.5874.$$

For the branch through $(0, 1)$, the corresponding endpoints are approximately

$$\boxed{x \approx \pm 3.3488.}$$

So the solution is valid on roughly

$$\boxed{-3.3488 < x < 3.3488.}$$

Summary

For separable equations,

$$\boxed{\frac{dy}{dx} = g(x)h(y)}$$

the central method is

$$\boxed{\frac{dy}{h(y)} = g(x)\,dx.}$$

- Check constant solutions before dividing by $h(y)$.
- Integrate to obtain an implicit or explicit solution.
- Use the initial condition to determine the constant.
- State the interval on which the solution is valid.