Chapter 2. First Order Differential Equations

Section 1. Linear Equations; Method of Integrating Factors

The class of equations

Standard linear form

$$\boxed{\frac{dy}{dt} + p(t)y = g(t)}$$

- "Linear" means y and y' appear only to the first power.
- The coefficient functions $p(t)$ and $g(t)$ may vary with t.
- Goal: multiply by $\mu(t)$ so that the left side is one derivative.

Where the integrating factor comes from

We want

$$\mu y' + \mu p(t) y = \frac{d}{dt}\left[\mu(t) y(t)\right] = \mu y' + \mu' y.$$

Thus we need

$$\mu' = p(t)\mu.$$

Solving the above equation gives

$$\boxed{\mu(t) = \exp\left(\int p(t)\, dt\right).}$$

Then

$$\frac{d}{dt}\left[\mu y\right] = \mu g(t).$$

General solution formula

Thus we have

$$\mu(t)y(t) = \int \mu(t)g(t)\,dt + C.$$

Equivalently, using a base point t_0,

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)\,ds + C\right].}$$

Comment:

Do not choose μ by guessing the solution; choose μ so that the product rule applies.

Example 1: a first integrating factor

Solve

$$y' + \frac{1}{2}y = \frac{1}{2}e^{t/3}.$$

Here $p(t) = 1/2$, so

$$\mu(t) = e^{t/2}.$$

Multiplying:

$$e^{t/2}y' + \frac{1}{2}e^{t/2}y = \frac{1}{2}e^{5t/6},$$

so

$$\frac{d}{dt}\left(e^{t/2}y\right) = \frac{1}{2}e^{5t/6}.$$

Example 1: solution and IVP

Integrate:

$$e^{t/2}y = \frac{3}{5}e^{5t/6} + C.$$

Therefore

$$\boxed{y = \frac{3}{5}e^{t/3} + Ce^{-t/2}.}$$

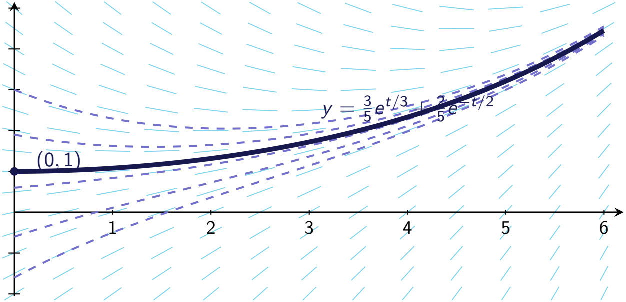
If the graph passes through $(t_0, y_0) = (0, 1)$, then

$$1 = \frac{3}{5} + C \quad \Rightarrow \quad C = \frac{2}{5}.$$

So

$$\boxed{y = \frac{3}{5}e^{t/3} + \frac{2}{5}e^{-t/2}.}$$

Example 1: qualitative picture



The direction field and several integral curves show the qualitative message: different C's give different curves, while the IVP selects the heavy curve through $(0, 1)$.

Example 2: first put in standard form

Solve

$$ty' + 2y = 4t^2, \qquad y(1) = 2.$$

Divide by t:

$$y' + \frac{2}{t}y = 4t.$$

Then

$$\mu(t) = e^{\int 2/t\, dt} = t^2 \quad (t > 0).$$

Multiply:

$$\frac{d}{dt}(t^2 y) = 4t^3.$$

Thus

$$t^2 y = t^4 + C, \qquad y = t^2 + \frac{C}{t^2}.$$

Example 2: interval matters

The initial condition $y(1) = 2$ gives

$$2 = 1 + C \quad \Rightarrow \quad C = 1.$$

Hence

$$\boxed{y = t^2 + \frac{1}{t^2}, \qquad t > 0.}$$

Why not cross $t = 0$?

The coefficient $2/t$ is discontinuous at $t = 0$, so the solution is naturally restricted to the interval containing the initial point: $(0, \infty)$.

Example 2: critical initial value

For a general condition $y(1) = y_0$,

$$y = t^2 + \frac{y_0 - 1}{t^2}, \qquad t > 0.$$

- If $y_0 > 1$, the solution goes to $+\infty$ as $t \to 0^+$.
- If $y_0 < 1$, the solution goes to $-\infty$ as $t \to 0^+$.
- If $y_0 = 1$, the solution is $y = t^2$, bounded at $t = 0$.

Example 3: integral form is acceptable

Solve

$$2y' + ty = 2, \qquad y(0) = 1.$$

Standard form:

$$y' + \frac{t}{2}y = 1.$$

Integrating factor:

$$\mu(t) = e^{t^2/4}.$$

Then

$$\frac{d}{dt}\left(e^{t^2/4}y\right) = e^{t^2/4}.$$

Example 3: solution with a non-elementary integral

Integrating from 0 to t,

$$e^{t^2/4} y = \int_0^t e^{s^2/4}\, ds + C.$$

The initial condition gives $C = 1$:

$$\boxed{y = e^{-t^2/4}\left(1 + \int_0^t e^{s^2/4}\, ds\right).}$$

Comment:

A solution may be exact even when an antiderivative cannot be written using elementary functions.

Summary

- Put first order linear equations into $y' + p(t)y = g(t)$.
- Use $\mu(t) = e^{\int p(t)dt}$.
- Then μy differentiates as a product.
- Initial conditions determine C and the interval of validity.
- Integral-form solutions are legitimate and useful.