Chapter 1. Introduction

Lecture 2. Solutions and Classification of Differential Equations

Review: Initial value problem

For the initial value problem:

$$\frac{dy}{dt} = ay - b, \qquad y(0) = y_0,$$

the solution is

$$\boxed{y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right) e^{at}.}$$

Interpretation:

One differential equation gives infinitely many integral curves; one initial condition selects one curve.

Example 1: falling object with linear drag

Assume downward velocity is positive. Newton's law gives

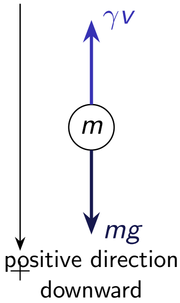
$$m\frac{dv}{dt} = mg - \gamma v.$$

For $m = 10$ kg, $\gamma = 2$ kg/s, $g = 9.8$ m/s^2,

$$\boxed{\frac{dv}{dt} = 9.8 - \frac{v}{5}.}$$

The equilibrium velocity is

$$9.8 - \frac{v}{5} = 0 \quad \Rightarrow \quad v_e = 49 \text{ m/s}.$$



Example 1: solve for velocity

Solve

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}, \qquad v(0) = 0.$$

Rewrite in the form $y' = ay - b$:

$$\frac{dv}{dt} = -\frac{1}{5}v + 9.8.$$

Thus $a = -\frac{1}{5}$, $b = -9.8$, and $b/a = 49$. Hence

$$v(t) = 49 + (0 - 49)e^{-t/5}.$$

So

$$\boxed{v(t) = 49\left(1 - e^{-t/5}\right).}$$

Example 1: from velocity to distance

Let $x(t)$ be the distance fallen. Since $v = dx/dt$,

$$\frac{dx}{dt} = 49\left(1 - e^{-t/5}\right).$$

Integrating,

$$x(t) = 49t + 245e^{-t/5} + C.$$

Because $x(0) = 0$,

$$0 = 245 + C \quad \Rightarrow \quad C = -245.$$

Therefore

$$\boxed{x(t) = 49t + 245e^{-t/5} - 245.}$$

Example 1: time of impact

If the object is dropped from height 300 m, then impact occurs when

$$x(T) = 300.$$

Thus

$$49T + 245e^{-T/5} - 245 = 300,$$

or

$$\boxed{49T + 245e^{-T/5} - 545 = 0.}$$

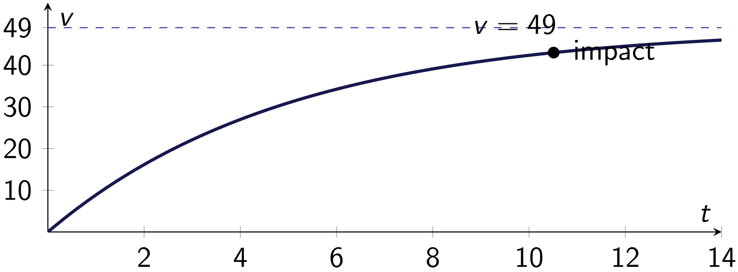
Numerically,

$$\boxed{T \approx 10.51 \text{ s}}.$$

The impact velocity is

$$v(T) = 49(1 - e^{-T/5}) \approx \boxed{43.01 \text{ m/s}}.$$

Example 1: velocity approaches terminal velocity



The velocity increases, but the drag force prevents indefinite acceleration.

Classification 1: ODE or PDE

Ordinary differential equation

The unknown function depends on one independent variable.

$$\frac{dy}{dt} = ay - b, \qquad y = y(t).$$

$$LQ'' + RQ' + \frac{1}{C}Q = E(t).$$

Partial differential equation

The unknown function depends on two or more independent variables.

$$\alpha^2 u_{xx} = u_t, \qquad u = u(x,t).$$

$$a^2 u_{xx} = u_{tt}.$$

Classification 2: single equation or system

Single unknown function

$$\frac{dy}{dt} = f(t, y).$$

System of differential equations

More than one unknown function is involved. For example, predator-prey equations:

$$\begin{cases} \dfrac{dx}{dt} = ax - \alpha xy, \\ \dfrac{dy}{dt} = -cy + \gamma xy. \end{cases}$$

Here $x(t)$ may represent prey and $y(t)$ may represent predators.

Classification 3: order

Definition

The order of a differential equation is the order of the highest derivative appearing in it.

- First order:

$$y' = 3 - 2y.$$

- Second order:

$$y'' + y = 0.$$

- Third order:

$$y''' + 2e^t y'' + yy' = t^4.$$

- General nth order ODE:

$$F(t, y, y', \dots, y^{(n)}) = 0.$$

Classification 4: linear or nonlinear

An nth order ODE is linear if it can be written as

$$\boxed{a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t).}$$

Linear means

- no products such as yy' or xy between unknown functions;
- no powers such as y^2, $(y')^2$;
- no nonlinear functions such as $\sin y$ or e^y.

What does it mean to be a solution?

For an *n*th order equation

$$y^{(n)} = f(t, y, y', \ldots, y^{(n-1)}),$$

a function $\phi(t)$ is a solution on $\alpha < t < \beta$ if

$$\phi, \phi', \ldots, \phi^{(n)} \text{ exist on } (\alpha, \beta),$$

and

$$\boxed{\phi^{(n)}(t) = f(t, \phi(t), \phi'(t), \ldots, \phi^{(n-1)}(t))}$$

for every t in the interval.

Summary

- Initial data choose one integral curve from a family.
- Equilibrium solutions help describe qualitative behavior.
- Differential equations are classified as ODE/PDE, single/system, by order, and by linearity.
- A proposed solution can always be checked by substitution.