Chapter 3. Second Order Linear Equations

Section 3.3. Complex Roots of the Characteristic Equation (II)

Recall from Class I

For the characteristic equation

$$ar^2 + br + c = 0, \qquad b^2 - 4ac < 0,$$

write the roots as $r = \lambda \pm i\mu$, $\mu > 0$.

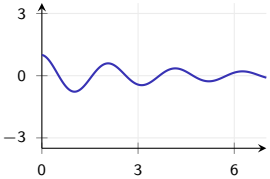
Real general solution

$$\boxed{y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t).}$$

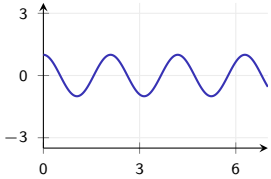
Three possible types of oscillation

Consider $y(t) = e^{\lambda t}\cos(3t)$ for different λ.

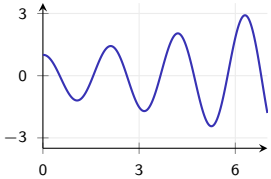
$\lambda < 0$: **decaying**



$\lambda = 0$: **constant**



$\lambda > 0$: **growing**



Main distinction

The sign of λ determines growth or decay; μ controls the oscillation rate.

Oscillation rate and exponential envelope

For

$$y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t),$$

the trigonometric part is periodic with

$$\boxed{\text{angular frequency } \mu, \qquad \text{period } T = \frac{2\pi}{\mu}.}$$

Let $R = \sqrt{c_1^2 + c_2^2}$. Since

$$|c_1 \cos \mu t + c_2 \sin \mu t| \leq R,$$

we obtain

$$\boxed{|y(t)| \leq Re^{\lambda t}.}$$

Graphing principle

The two exponential curves $y = \pm Re^{\lambda t}$ act as envelopes of the oscillation.

Example 2: characteristic roots

Solve the initial value problem

$$\boxed{16y'' - 8y' + 145y = 0}, \qquad y(0) = -2, \quad y'(0) = 1.$$

The characteristic equation is

$$16r^2 - 8r + 145 = 0.$$

Its discriminant is

$$(-8)^2 - 4(16)(145) = 64 - 9280 = -9216.$$

Hence

$$r = \frac{8 \pm 96i}{32} = \boxed{\frac{1}{4} \pm 3i}.$$

Real general solution

$$y(t) = e^{t/4}(c_1 \cos 3t + c_2 \sin 3t).$$

Example 2: the initial values

Use

$$y(t) = e^{t/4}(c_1 \cos 3t + c_2 \sin 3t).$$

From $y(0) = -2$ we get $c_1 = -2$. Differentiate and evaluate at $t = 0$:

$$y'(0) = \frac{1}{4}c_1 + 3c_2 = 1.$$

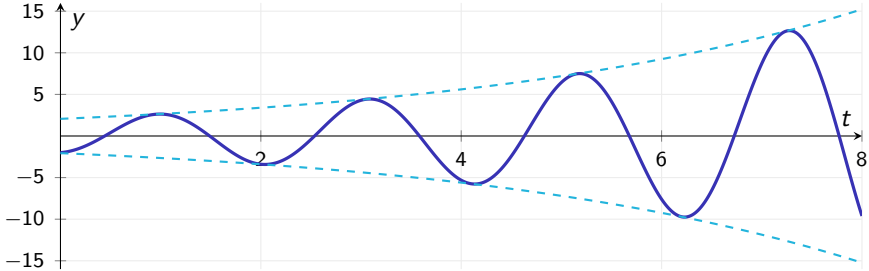
Thus

$$-\frac{1}{2} + 3c_2 = 1, \qquad c_2 = \frac{1}{2}.$$

The special solution

$$\boxed{y(t) = e^{t/4}\left(-2\cos 3t + \frac{1}{2}\sin 3t\right).}$$

Growing oscillation



Solid: solution. Dashed: envelopes $y = \pm\frac{\sqrt{17}}{2}e^{t/4}$.

Interpret Example 2

The solution is

$$y(t) = e^{t/4}\left(-2\cos 3t + \frac{1}{2}\sin 3t\right).$$

Two separate effects

- $\cos 3t$ and $\sin 3t$ cause oscillation.
- $e^{t/4}$ increases the size of successive oscillations.

$$\left|-2\cos 3t + \tfrac{1}{2}\sin 3t\right| \leq \frac{\sqrt{17}}{2}.$$

The curve repeatedly moves between its increasing positive and negative envelopes.

Example 3: pure oscillations

Solve

$$\boxed{y'' + 9y = 0.}$$

The characteristic equation is

$$r^2 + 9 = 0, \qquad r = \pm 3i.$$

Here $\lambda = 0$, $\mu = 3$, so

General solution

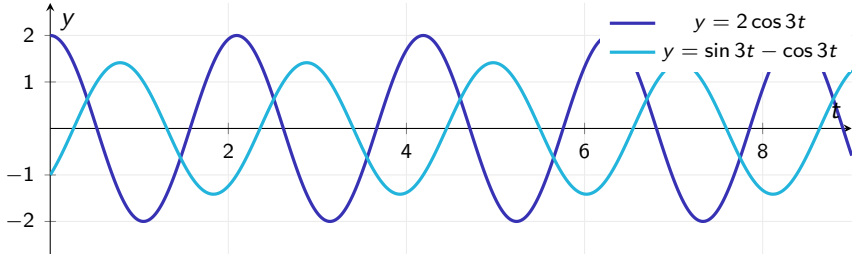
$$\boxed{y(t) = c_1 \cos 3t + c_2 \sin 3t.}$$

There is no growing or decaying exponential factor.

Graphical meaning

All nonzero solutions are sinusoidal, with constant amplitude and period $2\pi/3$.

Constant amplitude



Different initial values produce different amplitudes and phases, not different periods.

A link back to Section 3.2: Abel's theorem

The Wronskian of

$$u = e^{\lambda t}\cos \mu t, \qquad v = e^{\lambda t}\sin \mu t$$

is

$$\boxed{W(u, v)(t) = \mu e^{2\lambda t} \neq 0.}$$

Check this against Abel's formula: in standard form,

$$y'' + \frac{b}{a}y' + \frac{c}{a}y = 0.$$

Thus

$$W(t) = Ce^{-(b/a)t} = Ce^{2\lambda t} \qquad \left(\lambda = -\frac{b}{2a}\right).$$

Consistency check

Direct calculation agrees with Abel's theorem, with $C = \mu$.

Example 4: the initial value problem

Solve

$$y'' + 4y' + 5y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

The characteristic roots are

$$r^2 + 4r + 5 = 0, \qquad r = -2 \pm i.$$

Hence

$$y(t) = e^{-2t}(c_1 \cos t + c_2 \sin t).$$

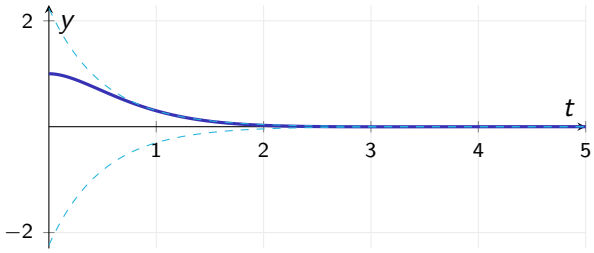
The data give

$$c_1 = 1, \qquad -2c_1 + c_2 = 0, \quad c_2 = 2.$$

Answer

$$\boxed{y(t) = e^{-2t}(\cos t + 2 \sin t).}$$

Example 4: sketch and predict



Properties

- $y(0) = 1$.
- $y'(0) = 0$.
- Periodic factors have period 2π.
- Exponential envelope decays rapidly.

Summary

Main formula

If $ar^2 + br + c = 0$ has roots $\lambda \pm i\mu$ with $\mu > 0$,

$$\boxed{y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t).}$$

- $\lambda < 0$: decaying oscillation; $\lambda = 0$: constant amplitude; $\lambda > 0$: growing oscillation.
- μ gives the oscillation period $2\pi/\mu$.
- Use initial values to find c_1 and c_2.