Chapter 3. Second Order Linear Equations

Section 3.3. Complex Roots of the Characteristic Equation (I)

Review: the imaginary unit i

By definition,

$$\boxed{i^2 = -1.}$$

The powers of i cycle:

$$i^0 = 1, \qquad i^1 = i, \qquad i^2 = -1, \qquad i^3 = -i, \qquad i^4 = 1.$$

Complex numbers

A complex number has the form $z = x + iy$, where x, y are real.

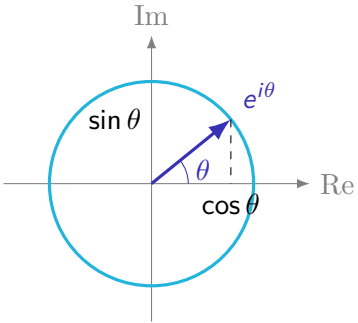
$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y.$$

A conjugate pair is $x + iy$ and $x - iy$.

Euler's formula

Fundamental identity

$$\boxed{e^{it} = \cos t + i \sin t.}$$



Point on the unit circle

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

- Real part: horizontal coordinate.
- Imaginary part: vertical coordinate.
- Magnitude: $|e^{i\theta}| = 1$.

Exponentials with general complex exponents

Let $r = \lambda + i\mu$ with $\mu \neq 0$.Then

$$\begin{aligned} e^{rt} &= e^{(\lambda+i\mu)t} = e^{\lambda t}e^{i\mu t} \\ &= \boxed{e^{\lambda t}\cos\mu t + ie^{\lambda t}\sin\mu t}. \end{aligned}$$

For its conjugate $\bar{r} = \lambda - i\mu$,

$$e^{\bar{r}t} = e^{\lambda t}\cos\mu t - ie^{\lambda t}\sin\mu t.$$

Conjugate pair

The real parts agree; the imaginary parts have opposite signs.

Complex exponentials solve the ODE

Euler's formula is compatible with

$$\boxed{\frac{d}{dt}e^{rt} = re^{rt}, \qquad r \in \mathbb{C}.}$$

Thus, for $y = e^{rt}$,

$$y' = re^{rt}, \qquad y'' = r^2 e^{rt}.$$

Substitute into $ay'' + by' + cy = 0$:

$$ay'' + by' + cy = (ar^2 + br + c)e^{rt}.$$

Conclusion

If r is a root of the characteristic equation, e^{rt} is a solution (possibly complex-valued).

Real and imaginary parts are solutions

Suppose $L[y] = ay'' + by' + cy$ with *real* a, b, c, and

$$y(t) = u(t) + iv(t), \qquad L[y] = 0.$$

Then

$$0 = L[u + iv] = L[u] + iL[v].$$

Taking real and imaginary parts gives

$$\boxed{L[u] = 0, \qquad L[v] = 0.}$$

Use this observation

From $e^{(\lambda + i\mu)t}$, obtain two *real* solutions

$$u(t) = e^{\lambda t} \cos \mu t, \quad v(t) = e^{\lambda t} \sin \mu t.$$

The textbook's sum-and-difference method

Set

$$y_1 = e^{(\lambda+i\mu)t}, \qquad y_2 = e^{(\lambda-i\mu)t}.$$

By superposition, their sum and difference are also solutions:

$$y_1 + y_2 = 2e^{\lambda t}\cos\mu t,$$

$$y_1 - y_2 = 2ie^{\lambda t}\sin\mu t.$$

Remove the nonzero constant multipliers 2 and $2i$ to obtain

$$\boxed{u = e^{\lambda t}\cos\mu t, \qquad v = e^{\lambda t}\sin\mu t.}$$

Check the Wronskian

Let $u = e^{\lambda t}\cos\mu t$ and $v = e^{\lambda t}\sin\mu t$.

$$u' = e^{\lambda t}(\lambda\cos\mu t - \mu\sin\mu t),$$
$$v' = e^{\lambda t}(\lambda\sin\mu t + \mu\cos\mu t).$$

Therefore

$$\begin{aligned} W(u,v) &= uv' - u'v \\ &= e^{2\lambda t}\mu(\cos^2\mu t + \sin^2\mu t) \\ &= \boxed{\mu e^{2\lambda t}}. \end{aligned}$$

Key result

$W(u,v) \neq 0$ for all real t because $\mu \neq 0$. Hence u, v form a fundamental set.

The main solution formula

Complex-root case

For

$$ay'' + by' + cy = 0, \quad b^2 - 4ac < 0,$$

with characteristic roots $\lambda \pm i\mu$ ($\mu > 0$),

$$\boxed{y(t) = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t).}$$

The constants c_1, c_2 are **real** for real initial conditions.

Example 1: characteristic equation

Solve

$$\boxed{y'' + y' + 9.25y = 0}, \qquad y(0) = 2, \quad y'(0) = 8.$$

The characteristic equation is

$$r^2 + r + 9.25 = 0.$$

Compute the discriminant:

$$D = 1 - 4(9.25) = -36.$$

Therefore

$$r = \frac{-1 \pm 6i}{2} = \boxed{-\frac{1}{2} \pm 3i}.$$

Identify the parameters

$\lambda = -1/2$ and $\mu = 3$.

Example 1: real fundamental set

The two complex exponentials are

$$y_1 = e^{(-1/2+3i)t}, \qquad y_2 = e^{(-1/2-3i)t}.$$

Their real-valued combinations are

$$u(t) = e^{-t/2}\cos 3t, \qquad v(t) = e^{-t/2}\sin 3t.$$

Their Wronskian is

$$W(u, v)(t) = 3e^{-t} \neq 0.$$

Real general solution

$$\boxed{y(t) = e^{-t/2}(c_1 \cos 3t + c_2 \sin 3t).}$$

Example 1: impose initial values

From $y(0) = 2$,

$$c_1 = 2.$$

For the derivative,

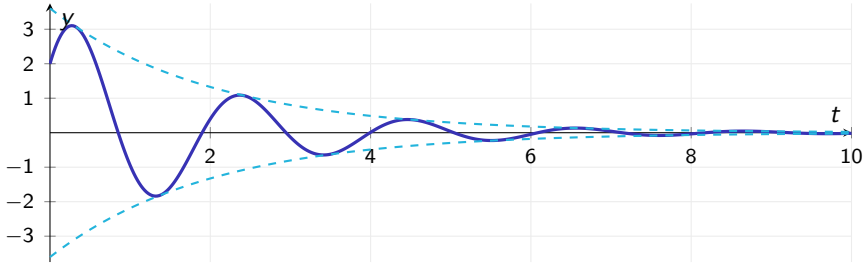
$$\begin{aligned} y'(t) = & -\tfrac{1}{2}e^{-t/2}(c_1 \cos 3t + c_2 \sin 3t) \\ & + 3e^{-t/2}(-c_1 \sin 3t + c_2 \cos 3t). \end{aligned}$$

Thus $y'(0) = -\frac{1}{2}c_1 + 3c_2 = 8$, so $c_2 = 3$.

Solution of the initial value problem

$$\boxed{y(t) = e^{-t/2}(2\cos 3t + 3\sin 3t).}$$

Decaying oscillation



Solid: solution. Dashed: envelopes $y = \pm\sqrt{13}\, e^{-t/2}$.

Why Example 1 oscillates and decays

We found

$$y(t) = e^{-t/2}(2\cos 3t + 3\sin 3t).$$

Oscillation

The sine and cosine terms oscillate with period $2\pi/3$.

Decay

The multiplier $e^{-t/2}$ decreases to zero as $t \to \infty$.

Indeed, since $|2\cos 3t + 3\sin 3t| \leq \sqrt{13}$,

$$|y(t)| \leq \sqrt{13}\, e^{-t/2} \longrightarrow 0.$$

Summary

Keep these three formulas

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta},$$

$$\boxed{r = \lambda \pm i\mu \quad \Longrightarrow \quad y = e^{\lambda t}(c_1 \cos\mu t + c_2 \sin\mu t)},$$

$$\boxed{W(e^{\lambda t}\cos\mu t, e^{\lambda t}\sin\mu t) = \mu e^{2\lambda t}.}$$